Non-Smooth Sustainable Development With Overshooting

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Non-Smooth Sustainable Development
With Overshooting

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Résumé / Abstract
Nous démontrons que, dans un modèle avec la substitution entre le capital et les ressources naturelles, le sentier du développement peut être non-monotone. Si l’on commence avec un niveau faible de capital et de ressources naturelles, le sentier optimal peut dépasser le niveau du capital de l’état stationnaire. La convergence s’effectue en temps fini.

Mots clés : développement soutenable, ressources naturelles renouvelables

We show that, in a model with substitutability between capital and resources, the path of sustainable development may be non-smooth, and may exhibit the overshooting property: starting from low levels of capital and resources, the economy may accumulate capital beyond its steady-state level, before converging to it in finite time.

Keywords: sustainable development, renewable resources

Codes JEL : C73, H41, D60

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1 Introduction

Since man-made capital and natural resources are substitutable inputs in the aggregate production function, a natural question that arises is how to optimally accumulate capital and manage the resource stock. The case where the natural resource stock is non-renewable has been studied by Solow under the maximin criterion, and Dasgupta and Heal (1979) and Pezzy and Withagen (1998) under the utilitarian criterion. Solow assumed a Cobb-Douglas production function, and showed that if the share of capital is greater than the share of natural resource, then a constant path of consumption is feasible, and along such a path, the man-made capital stock increases without bound. Dasgupta and Heal (1979) and Pezzy and Withagen (1998) showed that, under the utilitarian criterion, the man-made capital stock will reach a peak, and afterwards both stocks fall to zero asymptotically. Long and Katayama (2002) obtain similar results in a differential game model of common property resources and private capital accumulation.

In this paper, we study the optimal path for an economy that produces an output using a stock of capital and a resource input extracted from a stock of renewable natural resource. We retain the Solow-Dasgupta-Heal assumption that capital and resource are substitutable inputs in the production of the final good, but our model differs from theirs because the resource stock is renewable. We wish to find the optimal growth path of the economy under the utilitarian criterion. We show that there exists a unique steady state with positive consumption. We ask the following questions: (i) Can it be optimal to get to the steady state in finite time under the assumption that the utility function is strictly concave? (ii) Can finite-time approach paths to the steady state be smooth, in the sense that there are no jumps in the control variables? (iii) Are there non-smooth paths to the steady state?

The answers to the above questions are as follows.
There exists a set of initial conditions (which forms a one-dimensional manifold, i.e., a curve, in the state space) such that the approach path to the steady state takes a finite time, and is smooth. If the economy starts with a low resource stock, the path along the manifold toward the steady state involves gradual accumulation of the resource stock, and gradual running down of the capital stock toward its steady state level.

If the initial conditions are not on that one-dimensional manifold, then it may be optimal to get to some point on that manifold first, and then move along the manifold to get to the steady state. The path that gets to a point on the manifold is not smooth at the time it meets the manifold.

We show that starting from low levels of capital stock and resource stock, the optimal policy consists of three phases. In phase I, the planner builds up the stock of man-made capital above its steady state level, while the resource stock is kept below its steady state level. In phase II, the capital stock declines steadily, while the resource stock continues to grow, until the steady state is reached. In phase III, the economy stays at the steady state. Thus, our model exhibits the “overshooting” property.

Before proceeding, we would like to note that there are a number of articles that are somewhat related to our paper, where the authors discussed thr optimal use patterns for renewable resources and the sustainability of economies. Clark et al. (1979) provided a general formulation with irreversible investment. They focussed on irreversibility, and did not obtain an “overshooting” result. Among the relatively recent papers, Beltratti et al. (1998) addressed the problem of optimal use of renewable resources under a variety of assumptions about the objective of that economy (with the different types of the utility function.) They constructed a model in which a man-made capital stock and a renewable resource are used for production, and give a very general characterization of the paths which are optimal in various senses. Their basic model is similar to ours, however they focused on different issues.
We are not aware of any paper which examines the precise characteristics of steady state and of the approach paths to the steady state in a model with man-made capital and renewable resource.

2 The Model

We consider a continuous-time model. Let $N$ and $V$ denote the stock of man-made capital, and the stock of a renewable natural resource. Let $U$ denote the resource input. The output of the final good is

$$ Y = F(K, R) = \sqrt{KR} $$

Output can be consumed, or invested. Let $C$ denote consumption and $I$ denote investment. Then

$$ C = F(K, R) - I \quad (1) $$

Assume there is no depreciation of capital. Then

$$ \dot{K} = I \quad (2) $$

Let $\theta(S)$ be the natural growth function of the resource stock. We assume it has the shape of a tent. Specifically, we assume that there exists a stock level $\hat{S} > 0$ such that $\theta(S) = \omega S$ if $S < \hat{S}$, and $\theta(S) = \omega \hat{S} - \delta(S - \hat{S})$ for $S > \hat{S}$, where $\omega > 0$, $\delta > 0$. The net rate of growth of the resource stock is

$$ \dot{S} = \theta(S) - R \quad (3) $$

**Remark 1:** The function $\theta(S)$ has a kink at $\hat{S}$, so the derivative $\theta'(S)$ is not defined at $\hat{S}$. At that point, we define the generalised gradient of $\theta(S)$, denoted by $\partial \theta$, as the real interval $[-\delta, \omega]$, where $-\delta$ is the right-hand derivative, and $\omega$ is the left-hand derivative. When applying optimal control
theory, we must modify the equation for the shadow price of $S$ when $S$ is at $\hat{S}$. (This will be discussed in detail later.)

The consumption $C$ yields the utility

$$U(C) = \sqrt{C}$$

The objective of the planner is to maximize the integral of the discounted stream of utility:

$$\max \int_{0}^{\infty} \sqrt{Ce^{-\rho t}} \, dt$$

where we assume

$$0 < \rho < \omega$$

This assumption ensures that the optimal solution involves building the resource stock to the level $\hat{S}$.

The maximization is subject to

$$\dot{K} = \sqrt{KR} - C \quad (4)$$

$$\dot{S} = \theta(S) - R \quad (5)$$

with boundary conditions $K(0) = K_0 > 0$, $S(0) = S_0 > 0$, and

$$\lim_{t \to -\infty} K(t) > 0, \lim_{t \to -\infty} S(t) \geq 0$$

The set of positive stock levels is partitioned into two regions. Region I is the set of points $(S, K)$ such that $0 < S < \hat{S}$, and $K > 0$. Region II is the set of points $(S, K)$ such that $S \geq \hat{S}$, and $K > 0$.

We will show that there is no steady state in region I, and there is a unique steady state in region II. After that, we will show that in region I, there exists a unique one-dimensional manifold along which a smooth path converges to the steady state in region II. This manifold is downward sloping in the space $(S, K)$, so that along the smooth convergent path, the capital
stock falls and the resource stock rises. We then turn to region II and show that in that region, there exists also a unique one-dimensional manifold along which a smooth path converges to the steady state. We show that along this path, the capital stock rises and the resource stock falls.

From the above results, we infer that if the initial pair of stock levels \((S_0, K_0)\) does not belong to either of the two manifolds, the optimal path from such an initial point, if it converges to the steady state, must either involve a jump in some control variables, or an “overshooting” along the path.

### 3 Necessary conditions and steady state

#### 3.1 Necessary conditions in Region I

We define the current value Hamiltonian

\[
H = \sqrt{C} + \psi_1 \left[ \sqrt{KR} - C \right] + \psi_2 [\theta(S) - R]
\]

where \(\psi_1\) is the shadow price of man-made capital and \(\psi_2\) is the shadow price of the renewable resource.

The necessary conditions are

\[
\frac{\partial H}{\partial C} = \frac{1}{2\sqrt{C}} - \psi_1 = 0
\] (6)

\[
\frac{\partial H}{\partial R} = \frac{1}{2} \psi_1 \sqrt{\frac{K}{R}} - \psi_2 = 0
\] (7)

\[
\dot{\psi}_1 = \psi_1 (\rho - \frac{1}{2} \sqrt{\frac{R}{K}})
\] (8)

\[
\dot{\psi}_2 = \psi_2 (\rho - \omega)
\] (9)

Notice that \(\psi_1 > 0\) by (6). It follows that \(\psi_2 > 0\) by (7). So, in region I, \(\psi_2\) (the shadow price of the resource stock) is always falling because \(\rho < \omega\).
Thus we obtain the following result:

**Result 1:** There is no steady state in Region 1.

**Discussion:**

Here we make some remarks about the economic meaning of the necessary conditions.

From equations (7), (8) and (9) we get

\[(\rho - \omega) - (\rho - F_K) = \frac{\dot{\psi}_2}{\psi_2} - \frac{\dot{\psi}_1}{\psi_1} = \frac{1}{F_R} \frac{d(F_R)}{dt}\]

Hence

\[F_K = \omega + \frac{1}{F_R} \frac{d(F_R)}{dt} \quad (10a)\]

We may call equation (10a) the *Modified Hotelling Rule*: the rate of capital gain (rate of increase in the price of the extracted resource) plus the biological growth rate must be equated to the rate of interest on the capital good, $F_K$.

From (6) and (8), we get

\[\frac{\dot{C}}{2C} = F_K - \rho \quad (11)\]

which is the *Ramsey-Euler Rule*: the proportional rate of consumption growth, multiplied by the elasticity of marginal utility, must be equated to the difference between the rate of interest $F_K$ and the utility-discount rate, $\rho$.

It is convenient to define a new variable $x$:

\[x(t) = \frac{K(t)}{R(t)}\]

This variable is the capital/resource-input ratio, and is a measure of the *capital intensity* of the production process at time $t$.

Using (7) we get

\[x(t) = \left[\frac{2\psi_2(t)}{\psi_1(t)}\right]^2 \quad (12)\]
From this equation, we get

**Result 2:** $x(t)$ jumps at some time $t_1$ only if either $\psi_1$ or $\psi_2$ jumps at $t_1$.

**Discussion:** $\psi_2$ is continuous in Region I, but when $S(t)$ reaches $\hat{S}$ (which is in Region II) the kink in the growth function $\theta(S)$ may cause $\psi_2$ to jump.

### 3.2 The necessary conditions in Region II

The necessary conditions for Region II are a bit more complicated, because at the point $\hat{S}$ the function $\theta(S)$ is not differentiable. Thus we must deal with a “non-smooth” problem. For a general treatment of non-smooth optimal control problem see Clarke and Winter (1983), or Clarke (1983); here we follow the exposition in Docker et al (2000, pages 74-79).

Since $\theta(S)$ has a kink at $\hat{S}$, with left-hand derivative equal to $\omega > 0$ and right-hand derivative equal $-\delta$, the generalized gradient of $\theta(.)$ at $\hat{S}$ is defined as

$$\partial \theta(\hat{S}) = [-\delta, \omega]$$

The necessary conditions are

$$\frac{\partial H}{\partial C} = \frac{1}{2\sqrt{C}} - \psi_1 = 0 \quad (13)$$

$$\frac{\partial H}{\partial R} = \frac{1}{2} \psi_1 \sqrt{\frac{K}{R}} - \psi_2 = 0 = 0 \quad (14)$$

$$\dot{K} = \sqrt{KR} - C \quad (15)$$

$$\dot{\hat{S}} = \omega \hat{S} - \delta(S - \hat{S}) - R \text{ if } S > \hat{S} \quad (16)$$

$$\dot{\psi}_1 = \psi_1 (\rho - \frac{1}{2} \sqrt{\frac{R}{K}}) \quad (17)$$

and, from Docker et al. (2000, pages 74-79),

$$-(\dot{\psi}_2 - \rho \psi_2) \in [-\delta \psi_2, \omega \psi_2] \text{ if } S = \hat{S} \quad (18)$$
\[-(\dot{\psi}_2 - \rho \psi_2) = -\delta \psi_2 \text{ if } S > \hat{S}\] (19)

**Result 3**: There exists a unique steady state in Region II. The steady state resource stock is

\[S_{ss} = \hat{S}\]

and the steady state capital stock is

\[K_{ss} = K_{ss} = \omega \hat{S} \left[ \frac{\alpha}{\rho} \right]^{1/(1-\alpha)}\]

**Proof:**

Let us find the corresponding steady state values of other variables. From (16), at the steady state,

\[R_{ss} = \omega \hat{S}.\] (20)

From (17), at the steady state,

\[\alpha \left( \frac{K}{R} \right)^{\alpha-1} = \rho\] (21)

Thus

\[K_{ss} = \omega \hat{S} \left[ \frac{\alpha}{\rho} \right]^{1/(1-\alpha)}\] (22)

\[x_{ss} = \frac{K_{ss}}{R_{ss}} = \left[ \frac{\alpha}{\rho} \right]^{1/(1-\alpha)}\] (23)

Using (15), at the steady state

\[C_{ss} = \omega \hat{S} \left[ \frac{\alpha}{\rho} \right]^{\alpha/(1-\alpha)}\] (24)

Thus, from (13) and (24)

\[\psi_{ss1} = \left( \omega \hat{S} \right)^{-\gamma} \left[ \frac{\alpha}{\rho} \right]^{-\alpha \gamma/(1-\alpha)}\]
and, from (14)

$$
\psi_{ss2} = (\omega S)^{-\gamma} (1 - \alpha) \left[ \frac{\alpha}{\rho} \right]^{\alpha(1-\gamma)/(1-\alpha)}
$$

which is consistent with (18) because \( \rho \in [-\delta, \omega] \).

4 Dynamics in Region I

Since the steady state in region II is at the the boundary between the two regions, we are particularly interested in paths in Region I that converges to the steady state in region II, i.e. \( (S(t), K(t)) \rightarrow (S_{ss}, K_{ss}) \) in finite or infinite time. An important subclass of such convergent paths is called the paths of smooth convergent paths, by which we mean the control variables \( C(t) \) and \( R(t) \) do not jump (and hence \( x(t) \) does not jump).

4.1 The time path of capital/resource-input ratio in Region I

Lemma 1: In region I, the time path of the capital/resource-input ratio, \( x(t) \), satisfies the differential equation:

$$
-\frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} \dot{x} = -\omega
$$

(25)

It follows that:

1. if \( x_0 \) is optimally chosen, then

$$
\dot{x}(t) = \left( \left( \sqrt{x_0} - \frac{1}{2\omega} \right) e^{-\omega t} + \frac{1}{2\omega} \right)^2
$$

2. if at some time \( T \), the variable \( x \) takes the value \( x_T \), then

$$
\dot{x}(t) = \left( \left( \sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\omega(t-T)} + \frac{1}{2\omega} \right)^2 \equiv [g(t; x_T, T)]^2
$$

(26)
Proof: See Appendix A1.

Remark 2: If we impose the condition that at some time $T$ the variable $x(T)$ takes the following value (which is its steady state value in region II)

\[ x_T = \left( \frac{1}{2\rho} \right)^2 = x_{ss} \]  

then we can say something more definite about $x(t)$. See Lemma 2 below.

Lemma 2: If $x(t) \rightarrow x_T = x_{ss}$, then over the time interval $[0, T]$ the capital/resource-input ratio $x(t)$ decreases steadily.

Proof: From (26)

\[ \dot{x}(t) = 2g(t; x_T, T)g'(t; x_T, T) = -2\omega \left( \sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\omega(t-T)}g(t; x_T, T) < 0 \]

because

\[ \sqrt{x_T} = \frac{1}{2\rho} > \frac{1}{2\omega} \]

Remark 3: It can be shown (see Appendix A2) that if $x_T = x_{ss}$ then

\[ \frac{\dot{x}(t)}{x(t)} = -2\omega \left( 1 - \frac{1}{\left( \frac{\omega}{\rho} - 1 \right) e^{-\omega(t-T)} + 1} \right) \]

4.2 The time path of $\psi_1$

Lemma 3: In region I, the time path of $\psi_1$ is

\[ \psi_1(t) = \psi_{1T} \left( \frac{\sqrt{x_T}}{\left( \sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\rho(t-T)} + \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}} \right) \]

Proof: See Appendix A3.

Lemma 4: If $x(t) \rightarrow x_T = x_{ss}$ then over the time interval $[0, T]$ ,the shadow price of capital, $\psi_1(t)$, increases steadily.

Proof: See Appendix A4.
4.3 The time path of consumption in Region I

Lemma 5: In region I, the time path of consumption is

\[ C(t) = C_T \left( \frac{\sqrt{\bar{X}_T} - \frac{1}{2\omega}}{x_T} \right) e^{-\omega(t-T)} + \left( \frac{1}{2\omega} \right)^2 e^{-2(\rho-\omega)(t-T)} = C_T \left[ \frac{x(t)}{x_T} \right] e^{-2(\rho-\omega)(t-T)} \]

If \( x(t) \to x_T = x_{ss} \) over the time interval \([0, T]\), then consumption decreases steadily.

Proof: See Appendix A5.

4.4 The time path of extraction in Region I

Lemma 6: In region I, the time path of extraction is

\[ R(t) = \left( \frac{1}{2\rho} \right) \frac{C_T}{x_T} \exp(2\omega t - 2\omega T - 2\rho t + 2\rho T) + e^{2\omega t} E \]

where \( E \) satisfies

\[ R_T = \left( \frac{1}{2\rho} \right) \frac{C_T}{x_T} + e^{2\omega T} E \]

Thus, if (i) \( x(t) \to x_T = x_{ss} \), (ii) \( C(t) \to C_T = C_{ss} \) and (iii) \( R(t) \to R_T = R_{ss} \) then \( E = 0 \), and extraction will be rising steadily:

\[ \dot{R}(t) = 2(\omega - \rho) \omega S e^{2(\omega-\rho)(t-T)} > 0 \]

Proof: See Appendix 6.

4.5 The path of capital in region I

We now turn to the capital, we have

Lemma 7: Along the optimal path in Region I

\[ \frac{\dot{K}}{K} = x^{-\frac{1}{2}} - 2\rho \]
Along a smooth convergent path (i.e. \( x(t) \to x_T = x_{ss} \)) the capital stock \( K(t) \) falls steadily.

**Proof:** See Appendix A7.

**Lemma 8:** Along a smooth convergent path in Region I, there is a positive relationship between the time \( T \) and the initial stock \( K_0 \). It is given by

\[
K_0 = K(0) = \omega \tilde{S} e^{-2(\omega-\rho)T} \left( \left( \frac{1}{2\rho} - \frac{1}{2\omega} \right) e^{\omega T} + \frac{1}{2\omega} \right)^2 \tag{28}
\]

with

\[
\frac{dK_0}{dT} > 0 \tag{29}
\]

**Proof:** See Appendix A8.

### 4.6 The path of the resource stock in Region I

In Region I, the resource stock follows the law of motion

\[
\dot{S} = \omega S - R
\]

Thus we get

**Lemma 9:** Along a smooth convergent path in Region I (with \( R(t) \to R_T = R_{ss} = \omega \tilde{S} \)), there is a negative relationship between the time \( T \) and the initial resource stock \( S_0 \).

\[
S_0 = e^{-\omega T} \tilde{S} \left( \omega \frac{e^{-(\omega-2\rho)T} - 1}{-\omega + 2\rho} + 1 \right) \tag{30}
\]

with

\[
\frac{dS_0}{dT} < 0 \tag{31}
\]

**Proof:** See Appendix 9.

**PROPOSITION 1:** In Region I, the set of initial stock pairs \( (S, K) \) from which the optimal path is a smooth convergent path is the one dimensional
manifold defined by the two equations (28) and (30). This manifold has a negative slope in the space \((S, K)\).

**Proof:** Use (29) and (31):

\[
\frac{dK_0}{dS_0} < 0
\]

5 **Dynamics in Region II**

5.1 **The time path of capital/resource-input ratio in Region II:**

**Lemma 1b:** In region II, the time path of the capital/resource-input ratio, \(x(t)\), satisfies the differential equation:

\[
-\frac{1}{2} \frac{x^{-\frac{1}{2}}}{x} + \frac{1}{2} \frac{\dot{x}}{x} = \delta
\]

(32)

It follows that:

1. if \(x_0\) is optimally chosen, then

\[
x(t) = x(t) = \left( \sqrt{x_0} + \frac{1}{2\delta} \right)^2
\]

2. if at some time \(T\), the variable \(x\) takes the value \(x_T\), then

\[
x(t) = \left( \left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{\delta(t-T)} - \frac{1}{2\delta} \right)^2 \equiv [f(t; x_T, T)]^2
\]

**Proof:** See Appendix 10

**Lemma 2:** In Region II, over the time interval \([0, T]\) the capital/resource-input ratio \(x(t)\) increases steadily.

**Proof:** from (32)

\[
-\frac{1}{2} + \frac{1}{2} \frac{\dot{x}}{\sqrt{x}} = \delta x^\frac{1}{2}
\]
so $\dot{x}$ must be positive.

**Remark 3b:** We retrieve the results of Region I if we substitute $\delta$ by $-\omega$.

### 5.2 The path of $\psi_1$ in region II

**Lemma 3b:** In region I, the time path of $\psi_1$ is

$$\psi_1(t) = \frac{\psi_1(T)e^{(\delta+\rho)(t-T)}}{((1 + \frac{\rho}{\delta})e^{\delta(t-T)} - \frac{\rho}{\delta})}$$

**Proof:** See Appendix 11.

### 5.3 The time path of consumption in Region II

**Lemma 5b:** In region II, the time path of consumption is

$$c(t) = C_T \left( \left( \frac{\sqrt{x_T} + \frac{1}{2\gamma}}{\sqrt{x_T}} \right) e^{\delta(t-T)} - \frac{1}{2\gamma} \right) e^{-(\delta+\rho)(t-T)} = C_T \left[ \frac{x(t)}{x_T} \right] e^{-2(\rho+\delta)(t-T)}$$

If $x(t) \to x_T = x_{ss}$ over the time interval $[0, T]$, then consumption decreases steadily.

**Proof:** See Appendix 12.

### 5.4 The path of extraction in region II

**Lemma 6b:** In region I, the time path of extraction satisfies the differential equation

$$\dot{R} = -2\delta R - \frac{C_T e^{-2(\delta+\rho)(t-T)}}{x_T}$$

Thus, if (i) $x(t) \to x_T = x_{ss}$, (ii) $C(t) \to C_T = C_{ss}$ and (iii) $R(t) \to R_T = R_{ss}$

$$R(t) = \omega \hat{S} e^{-2(\delta+\rho)(t-T)}$$
\[ \dot{R}(t) < 0 \]

**Proof:** See Appendix 13

### 5.5 The path of capital in region II

We now turn to the capital, we have

**Lemma 7b:** Along the optimal path in Region II

\[ \frac{\dot{K}}{K} = x^{-\frac{1}{2}} - 2\rho \]

Along a smooth convergent path (i.e. \( x(t) \to x_T = x_{ss} \)) the capital stock \( K(t) \) falls steadily.

**Proof:** See Appendix 14.

**Lemma 8b:** Along a smooth convergent path in Region II, there is a positive relationship between the time \( T \) and the initial stock \( K_0 \). It is given by

\[ K_0 = K(0) = \omega S \left( \left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{-\delta T} - \frac{1}{2\delta} \right) e^{(\delta + \rho)T} \]

**Proof:** See Appendix 15.

### 5.6 The path of the resource stock in Region II

In region II we have

\[ \dot{S} = \omega \dot{S} - \delta (S - \dot{S}) - R \]

Substituting \( R \) gives

\[ \dot{S} = \omega \dot{S} - \delta (S - \dot{S}) - \omega S e^{-2(\delta + \rho)(t-T)} \]

\[ \dot{S} + \delta S = \omega \dot{S} + \delta \dot{S} - \omega e^{-2(\delta + \rho)(t-T)} \]

\[ \dot{S} + \delta S = \dot{S} (\omega + \delta - \omega e^{-2(\delta + \rho)(t-T)}) \]
Note that
\[ \dot{S}(T) + \delta \dot{S} = \delta \dot{S} \]
and thus
\[ \dot{S}(T) = 0 \]

We now solve for the path of the resource stock
\[ \begin{align*}
S' &= A \left( \omega + \delta - \omega e^{-2(\delta + \rho)(t-T)} \right) - \delta S \\
S(T) &= A
\end{align*} \]

The exact solution is:
\[ S(t) = \omega \frac{A}{\delta + \delta^2 - 2\rho} \exp \left( -2\delta t + 2\delta T - 2\rho t + 2\rho T \right) - 2e^{-\delta t} \omega \frac{A}{e^{-\delta T} \delta (\delta + 2\rho)} \]

we check that \[ S(T) = \omega \frac{S_0}{\delta + \delta^2 - 2\rho} - 2\omega \frac{S_0}{\delta (\delta + 2\rho)} = \hat{S} \]
Moreover
\[ \dot{S}(t) = \omega \frac{S_0}{\delta + 2\rho} e^{-2(\delta + \rho)(t-T)} - 2e^{-\delta(t-T)} \omega \frac{S_0}{\delta (\delta + 2\rho)} \]
\[ \dot{S}(t) = 2\omega \frac{S_0}{\delta + 2\rho} e^{-\delta(t-T)} \frac{\delta + \rho}{(\delta + 2\rho)} \left( -e^{-(\delta + 2\rho)(t-T)} + 1 \right) < 0 \]

There exists a smooth path reaching \( \hat{S} \) at \( T \) if \( S_0 \) satisfies
\[ S_0 = S(0) = \omega \frac{S_0}{\delta + \delta^2 - 2\rho} + \omega \frac{\hat{S}}{\delta + 2\rho} e^{2(\delta + \rho)T} - 2e^{\delta T} \omega \frac{S_0}{\delta (\delta + 2\rho)} \]
\[ \frac{dS_0}{dT} = \omega \frac{2(\delta + \rho)\hat{S}}{\delta + 2\rho} e^{2(\delta + \rho)T} - 2e^{\delta T} \omega \frac{\delta + \rho}{(\delta + 2\rho)} \]
\[ \frac{dS_0}{dT} = 2e^{\delta T} \omega \frac{\delta + \rho}{(\delta + 2\rho)} \left( e^{(\delta + 2\rho)T} - 1 \right) > 0 \]
In Region II we also have
\[ \frac{dS_0}{dT} > 0 \]
and
\[ \frac{dK_0}{dT} < 0 \]
so
\[ \frac{dK_0}{dS_0} < 0 \]

In Both region I and region II we have
\[ \frac{dK_0}{dS_0} < 0 \]

This implies either overshooting or jump in the control paths.

6 Concluding Remarks

We have been able to show that the path to a steady state may exhibit
the overshooting property. The economy accumulate capital to some level
much higher than its steady-state level, before running it down. This is
because when the renewable resource is still at a low level, more output can
be generated by accumulating capital, while using the resource sparingly.
When a sufficient large level of resource has been achieved, it becomes more
efficient to use more resource, and less capital, in the production process.

Our model displays two additional features: it takes a finite time to get to
the steady state, and the paths to the steady state is generally non-smooth,
unless the economy happens to have a combination of stock levels that lies
on the smooth one-dimensional manifold.

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APPENDIX 1:

Proof of Lemma 1

Step 1:

We first show that $x$ satisfies the following differential equation. This is shown from the necessary conditions (??), (6) and (8),

$$\frac{\partial H}{\partial C} = \frac{1}{2\sqrt{C}} - \psi_1 = 0 \quad (33)$$

$$\frac{1}{2\sqrt{C}} = \psi_1 \quad (34)$$

$$-\frac{1}{2} \ln C = \ln \psi_1 + \ln 2 \quad (35)$$

$$-\frac{\dot{C}}{2C} = \frac{\dot{\psi}_1}{\psi_1} \quad (36)$$

but

$$\frac{\dot{\psi}_1}{\psi_1} = \rho - \frac{1}{2} x^{-\frac{1}{2}} \quad (37)$$

so we have

$$-\frac{\dot{C}}{2C} = \frac{\dot{\psi}_1}{\psi_1} = \rho - \frac{1}{2} x^{-\frac{1}{2}} \quad (38)$$

We get the relationship between $\psi_1$ and $x$

$$\frac{\partial H}{\partial R} = \frac{1}{2} \psi_1 \sqrt{\frac{K}{R}} - \psi_2 = 0 \quad (39)$$

or

$$\frac{1}{2} \psi_1 \sqrt{x} - \psi_2 = 0 \quad (40)$$

$$-\ln 2 + \ln \psi_1 + \frac{1}{2} \ln x - \psi_2 = 0 \quad (41)$$

so that

$$\frac{\dot{\psi}_2}{\psi_2} = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2} \frac{x}{\dot{x}} \quad (42)$$

and

$$\rho - \omega = \rho - \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} \frac{x}{\dot{x}} \quad (43)$$
or
\[
-\omega = -\frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}}
\]  

(44)

This ends Step 1.

**Step 2:** Solving for \( x(t) \):

We have
\[
-\omega = -\frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} \frac{\dot{x}}{x}
\]

(45)
multiplying each side by \( \sqrt{x} \) gives

\[
-\frac{1}{2} + \frac{1}{2} \frac{\dot{x}}{\sqrt{x}} = -\omega x^{-\frac{1}{2}}
\]

(46)

let \( y \equiv \sqrt{x} \)

\[
-\frac{1}{2} + \frac{\dot{y}}{y} = -\omega y
\]

(47)

the solution can be written in two forms:

\[
y(t) = \left(y_0 - \frac{1}{2}\right) e^{-\omega t} + \frac{1}{2\omega}
\]

or

\[
y(t) = \left(y_T - \frac{1}{2\omega}\right) e^{-\omega (t-T)} + \frac{1}{2\omega}
\]

where \( y_0 = y(0) \) or \( y_T = y(T) \) and therefore we have

\[
x(t) = \left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega (t-T)} + \frac{1}{2\omega}\right)^2
\]

(48)

**APPENDIX 2:**

If \( x_T = x_{ss} \) then

\[
\frac{\ddot{x}(t)}{x(t)} = -2\omega \left(1 - \frac{1/2\omega}{\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega (t-T)} + \frac{1}{2\omega}}\right)
\]

\[
\frac{\ddot{x}(t)}{x(t)} = -2\omega \left(1 - \frac{1/2\omega}{\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega (t-T)} + \frac{1}{2\omega}}\right)
\]

20
\[
\frac{\dot{x}(t)}{x(t)} = -2\omega \left(1 - \frac{1/2\omega}{\left(\sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\omega(t-T)} + \frac{1}{2\omega}}\right)
\]

\[
\frac{\dot{x}(t)}{x(t)} = -2\omega \left(1 - \frac{1/2\omega}{\left(\frac{1}{2\rho} - \frac{1}{2\omega} \right) e^{-\omega(t-T)} + \frac{1}{2\omega}}\right)
\]

\[
\frac{\dot{x}(t)}{x(t)} = -2\omega \left(1 - \frac{1}{\left(\frac{\rho}{\omega} - 1\right) e^{-\omega(t-T)} + 1}\right)
\]

so since \(\left(\frac{\omega}{\rho} - 1\right) > 0\) then \(\frac{\dot{x}(t)}{x(t)} < 0\).

**APPENDIX 3: Proof of Lemma 3.**

We can solve for \(\psi_1\) from (12)

\[
\frac{\dot{\psi}_2}{\psi_2} = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2} \frac{\dot{x}}{x} \tag{49}
\]

with

\[
\dot{\psi}_2 = \psi_2 (\rho - \omega) \tag{50}
\]

so

\[
\frac{\dot{\psi}_1}{\psi_1} = \rho - \omega - \frac{1}{2} \frac{\dot{x}}{x}
\]

integrating gives

\[
\ln \frac{\psi_1(t)}{\psi_1(T)} = (\rho - \omega) (t - T) - \ln \sqrt{\frac{x(t)}{x(T)}}
\]

or

\[
\psi_1(t) = \psi_{1T} \frac{\sqrt{x_T}}{\sqrt{x(t)}} e^{(\rho-\omega)(t-T)}
\]

\[
\psi_1(t) = \psi_{1T} \left(\left(\sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right) e^{(\rho-\omega)(t-T)}
\]

\[
\psi_1(t) = \psi_{1T} \left(\left(\sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\rho(t-T)} + \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}\right)
\]

21
**APPENDIX 4: Proof of Lemma 3**

The denominator is \( D(t) = \left( \sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\rho(t-T)} + \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)} \) is such that

\[
D'(t) = -\rho \left( \sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\rho(t-T)} - (\rho - \omega) \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}
\]

\[
D'(t) = -\rho \left( \frac{1}{2\rho} - \frac{1}{2\omega} \right) e^{-\rho(t-T)} - (\rho - \omega) \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}
\]

\[
D'(t) = \frac{1}{2\omega} \left( -\rho \left( \frac{\omega}{\rho} - 1 \right) e^{-\rho(t-T)} - (\rho - \omega) e^{-(\rho-\omega)(t-T)} \right)
\]

\[
D'(t) = \frac{1}{2\omega} ((\rho - \omega) e^{-\rho(t-T)} - (\rho - \omega) e^{-(\rho-\omega)(t-T)})
\]

\[
D'(t) = \frac{1}{2\omega} (\rho - \omega) e^{-\rho(t-T)} (1 - e^{\omega(t-T)}) < 0
\]

since \( \rho < \omega \). So

\[ \dot{\psi}_1(t) > 0 \]

**APPENDIX 5: Proof of Lemma 5**

\[
\frac{1}{2\sqrt{C}} - \psi_1 = 0
\]

or

\[
\frac{1}{2\psi_1} = \sqrt{C}
\]

or

\[
\left( \frac{1}{2\psi_1} \right)^2 = C
\]

that is

\[
C(t) = \frac{1}{\left( 2\psi_1 T \left( \frac{\sqrt{x_T}}{2\omega} e^{-\omega(t-T)} + \frac{1}{2\omega} \right) e^{(\rho-\omega)(t-T)} \right)^2}
\]

\[
C(t) = \frac{\left( \frac{\sqrt{x_T}}{2\omega} e^{-\omega(t-T)} + \frac{1}{2\omega} \right)^2}{x_T \left( 2\psi_1 T \right)^2} e^{-2(\rho-\omega)(t-T)}
\]

22
\[ C(t) = C_T \left( \frac{(\sqrt{x_T} - \frac{1}{2\omega}) e^{-\omega(t-T)} + \frac{1}{2\omega})^2}{x_T} e^{-2(\rho-\omega)(t-T)} \right) = C_T \left[ \frac{x(t)}{x_T} \right] e^{-2(\rho-\omega)(t-T)} \]

The evolution of the consumption path is given by

\[ \frac{\dot{C}}{C} = -2 \frac{\dot{\psi}_1}{\psi_1} \]

If \( x(t) \to x_T = x_{ss} \) over the time interval \([0, T]\) the \( C \) falls steadily because \( \psi_1 \) rises steadily.

**APPENDIX 6**

From the definition of \( x = K/R \), we have

\[ \dot{K} = \dot{R} x + R \dot{x} \]

and

\[ \dot{K} = \sqrt{KR} - C = R\sqrt{x} - C \]

so

\[ \dot{R} x + R \dot{x} = R\sqrt{x} - C \]

or

\[ \dot{R} = R \left( \frac{1}{\sqrt{x}} - \frac{\dot{x}}{x} \right) - \frac{C}{x} \]

using (25) yields

\[ -\omega = -\frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} \frac{\dot{x}}{x} \]

so

\[ \dot{R} = 2\omega R - \frac{C}{x} \]

where

\[ C(t) = C_T \left( \frac{(\sqrt{x_T} - \frac{1}{2\omega}) e^{-\omega(t-T)} + \frac{1}{2\omega})^2}{x_T} e^{-2(\rho-\omega)(t-T)} \right) \]
and
\[ x(t) = \left( \left( \sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\omega(t-T)} + \frac{1}{2\omega} \right)^2 \]
so
\[ \dot{R} = 2\omega R - \frac{C_T ((\sqrt{x_T} - \frac{1}{2\omega}) e^{-\omega(t-T)} + \frac{1}{2\omega})^2 e^{2(\rho-\omega)(t-T)}}{((\sqrt{x_T} - \frac{1}{2\omega}) e^{-\omega(t-T)} + \frac{1}{2\omega})^2} \] (55)

Hence
\[ \dot{R} = 2\omega R - \frac{C_T e^{-2(\rho-\omega)(t-T)}}{x_T} \] (56)

The solution is
\[ R(t) = \frac{1}{2} \frac{C_T}{x_T\rho} \exp\left(2\omega t - 2\omega T - 2\rho T + 2\rho T\right) + e^{2\omega t} E \]

and
\[ R_T = \frac{C_T}{x_T 2\rho} + e^{2\omega T} E \]
with
\[ x_T = \frac{K_{ss}}{R_{ss}} = \left( \frac{1}{2\rho} \right)^2 \]
\[ C_T = \omega \hat{S} \left[ \frac{1}{2\rho} \right] \] (57)

Now
\[ R_T = \omega \hat{S} \] (58)
so
\[ E = \left( R_T - \frac{C_T}{x_T 2\rho} \right) e^{-2\omega T} = 0 \]
and
\[ R(t) = \omega \hat{S} e^{2(\omega-\rho)(t-T)} \] (59)
\[ \dot{R}(t) = 2(\omega - \rho) \omega \hat{S} e^{2(\omega-\rho)(t-T)} > 0 \]

APPENDIX 7: Proof of Lemma 7

\[ K = xR \]
\[
\frac{\dot{K}}{K} = \frac{\dot{x}}{x} + \frac{\dot{R}}{R} = \frac{\dot{x}}{x} + 2(\omega - \rho)
\]
\[
\frac{\dot{K}}{K} = -2\omega + x^{-\frac{1}{2}} + 2(\omega - \rho)
\]
\[
\frac{\dot{K}}{K} = x^{-\frac{1}{2}} - 2\rho
\]

since \(\dot{x} < 0\) we have \(\frac{dx^{-\frac{1}{2}}}{dt} > 0\) with \((x(T))^{-\frac{1}{2}} = 2\) and therefore \(x^{-\frac{1}{2}} - 2\rho < 0\) for all \(t < T\) and thus
\[
\frac{\dot{K}}{K} = x^{-\frac{1}{2}} - 2\rho < 0
\]

**APPENDIX 8: Proof of Lemma 8**

Substituting for \(x\) and \(R\) gives

\[
K = xR = \omega \widehat{S}e^{2(\omega - \rho)(t-T)} \left( \left( \sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\omega(t-T)} + \frac{1}{2\omega} \right)^2
\]

at time \(t = 0\) we have

\[
K_0 = K(0) = \omega \widehat{S}e^{-2(\omega - \rho)T} \left( \left( \frac{1}{2\rho} - \frac{1}{2\omega} \right) e^{\omega T} + \frac{1}{2\omega} \right)^2
\]

\[
\frac{dK_0}{dT} = \omega \widehat{S}e^{-2(\omega - \rho)T} \left( \left( \frac{1}{2\rho} - \frac{1}{2\omega} \right) e^{\omega T} + \frac{1}{2\omega} \right)^2
\]

\[
\frac{dK_0}{dT} = \omega \widehat{S} \frac{d}{dT} \left( \left( \frac{1}{2\rho} - \frac{1}{2\omega} \right) e^{\omega T} + \frac{1}{2\omega} \right)^2
\]

Let \(f(T) = e^{-(\omega - \rho)T} \left( \left( \frac{1}{2\rho} - \frac{1}{2\omega} \right) e^{\omega T} + \frac{1}{2\omega} \right)\) we have

\[
f(T) = \left( \frac{1}{2\rho} - \frac{1}{2\omega} \right) e^{\rho T} + \frac{1}{2\omega} e^{-(\omega - \rho)T}
\]

\[
f'(T) = \rho \left( \frac{1}{2\rho} - \frac{1}{2\omega} \right) e^{\rho T} - (\omega - \rho) \frac{1}{2\omega} e^{-(\omega - \rho)T}
\]

\[
f'(T) = \frac{1}{2\omega} \left( \rho \left( \frac{\omega}{\rho} - 1 \right) e^{\rho T} - (\omega - \rho) e^{-(\omega - \rho)T} \right)
\]
\[ f'(T) = \frac{1}{2\omega} (\omega - \rho) e^{\rho T} (1 - e^{-\omega T}) > 0 \]

So

\[ \frac{dK_0}{dT} = \omega\hat{S} f'(T) f(T) > 0 \]

**APPENDIX 9: Proof of Lemma 9.**

Substituting \( R \) to get

\[ \dot{S} = \omega S - \omega\hat{S} e^{2(\omega-\rho)(t-T)} \]

The exact solution is:

\[
\begin{align*}
S(t) &= \hat{S} \left( \frac{\omega e^{2(\omega-\rho)(t-T)} + 2e^{\omega(t-T)} - \omega + \rho}{(-\omega + 2\rho)} \right) \\
\dot{S}(t) &= \hat{S} \left[ \frac{2(\omega - \rho) e^{-\omega(T-t)} - \omega e^{-2(\omega-\rho)(T-t)}}{(2(\omega - \rho) - \omega)} \right] \\
\dot{S}(t) &= 2\omega (\omega - \rho) e^{\omega(t-T)} \hat{S} \left( \frac{1}{-\omega + 2\rho} e^{(\omega-2\rho)(t-T)} - \frac{1}{(-\omega + 2\rho)} \right) \\
\dot{S}(t) &= 2\omega (\omega - \rho) e^{\omega(t-T)} \hat{S} \left( \frac{e^{(\omega-2\rho)(t-T)} - 1}{(-\omega + 2\rho)} \right) > 0
\end{align*}
\]

The initial stock must be

\[
S_0 = S(0) = \hat{S} \left( \frac{\omega}{-\omega + 2\rho} e^{2(\omega-\rho)T} + 2e^{-\omega T} - \omega + \rho \right) \\
S_0 = e^{-\omega T} \hat{S} \left( \frac{e^{-(\omega-2\rho)T} - 1}{-\omega + 2\rho} + 1 \right) \\
\frac{dS_0}{dT} = \hat{S} \left( -2(\omega - \rho) \omega e^{-2(\omega-\rho)T} + 2e^{-\omega T} - \omega + \rho \right) \\
\frac{dS_0}{dT} = 2(\omega - \rho) \omega \hat{S} e^{-\omega T} \left( \frac{1 - e^{-(\omega-2\rho)T}}{-\omega + 2\rho} \right) < 0
\]
So we have
\[ \frac{dS_0}{dT} = 2 (\omega - \rho) \omega \hat{S} e^{-\omega T} \left( \frac{1 - e^{-(\omega - 2\rho)T}}{-\omega + 2\rho} \right) < 0 \]

and
\[ \frac{dK_0}{dT} > 0 \]

and therefore in Region I:
\[ \frac{dK_0}{dS_0} < 0 \]

**APPENDIX 10. Proof of Lemma 1b.**

**Step 1:**
We first show that \( x \) satisfies the following differential equation
\[ -\frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} \frac{\dot{x}}{x} = \delta \]  
(60)

This is shown from the necessary conditions (??), (6) and (8),
\[ \frac{\partial H}{\partial C} = \frac{1}{2\sqrt{C}} - \psi_1 = 0 \]  
(61)
\[ \frac{1}{2\sqrt{C}} = \psi_1 \]  
(62)
\[ -\frac{1}{2} \ln C = \ln \psi_1 + \ln 2 \]  
(63)
\[ -\frac{\dot{C}}{2C} = \frac{\dot{\psi}_1}{\psi_1} \]  
(64)

but we know from the necessary conditions
\[ \frac{\dot{\psi}_1}{\psi_1} = \rho - \frac{1}{2} x^{-\frac{1}{2}} \]  
(65)

so we have
\[ -\frac{\dot{C}}{2C} = \frac{\dot{\psi}_1}{\psi_1} = \rho - \frac{1}{2} x^{-\frac{1}{2}} \]  
(66)
The relationship between $\psi_1$ and $x$ is from

\[
\frac{\partial H}{\partial R} = \frac{1}{2} \psi_1 \sqrt{\frac{K}{R}} - \psi_2 = 0
\]

(67)

or

\[
\frac{1}{2} \psi_1 \sqrt{x} - \psi_2 = 0
\]

(68)

\[- \ln 2 + \ln \psi_1 + \frac{1}{2} \ln x - \psi_2 = 0
\]

(69)

so that

\[
\frac{\dot{\psi}_2}{\psi_2} = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2} \frac{\dot{x}}{x}
\]

(70)

and from the necessary conditions we have

\[
\dot{\psi}_2 = \psi_2 (\delta + \rho)
\]

(71)

so we have

\[
\delta + \rho = \rho - \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} \frac{\dot{x}}{x}
\]

(72)

or

\[
\delta = - \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} \frac{\dot{x}}{x}
\]

(73)

This ends Step 1

**Step 2:** Solving for $x(t)$:

We have

\[
\delta = - \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} \frac{\dot{x}}{x}
\]

(74)

multiplying each side by $\sqrt{x}$ gives

\[
- \frac{1}{2} + \frac{1}{2} \frac{\dot{x}}{\sqrt{x}} = \delta x^{\frac{1}{2}}
\]

(75)

let $y \equiv \sqrt{x}$

\[
- \frac{1}{2} + \dot{y} = \delta y
\]

(76)
the solution can be written in two forms:

\[
y(t) = \left( y_0 + \frac{1}{2\delta} \right) e^{\delta t} - \frac{1}{2\delta}
\]

or

\[
y(t) = \left( y_T + \frac{1}{2\delta} \right) e^{\delta(t-T)} - \frac{1}{2\delta}
\]

where \( y_0 = y(0) \) or \( y_T = y(T) \) and therefore we have

\[
x(t) = \left( \left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{\delta(t-T)} - \frac{1}{2\delta} \right)^2
\]

or

\[
x(t) = \left( \left( \sqrt{x_0} + \frac{1}{2\delta} \right) e^{\delta t} - \frac{1}{2\delta} \right)^2
\]

This ends Step 2.

**APPENDIX 11: Proof of Lemma 3b**

We have

\[
\frac{\dot{\psi}_2}{\psi_2} = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2} \frac{\dot{x}}{x}
\]

\[
\delta + \rho = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2} \frac{\dot{x}}{x}
\]

\[
\delta + \rho = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2} \frac{\dot{x}}{x} + \frac{1}{2} \frac{\sqrt{x_T + \frac{1}{2\delta}}}{\sqrt{x_T}} e^{\delta(t-T)} \left( \left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{\delta(t-T)} - \frac{1}{2\delta} \right)
\]

\[
\delta + \rho = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2} \frac{\dot{x}}{x} + \frac{\delta}{\left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{\delta(t-T)} - \frac{1}{2\delta}}
\]

\[
(\delta + \rho) (t - T) = \ln \frac{\psi_1(t)}{\psi_1(T)} + \ln \left( \frac{\left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{\delta(t-T)} - \frac{1}{2\delta}}{\sqrt{x_T}} \right)
\]

\[
e^{(\delta + \rho)(t-T)} = \frac{\psi_1(t)}{\psi_1(T)} \left( \frac{\left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{\delta(t-T)} - \frac{1}{2\delta}}{\sqrt{x_T}} \right)
\]

\[
\frac{\sqrt{x_T} \psi_1(T) e^{(\delta + \rho)(t-T)}}{\left( \left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{\delta(t-T)} - \frac{1}{2\delta} \right)} = \psi_1(t)
\]
\[
\frac{1}{2\rho} \psi_1(T) e^{(\delta + \rho)(t-T)} = \psi_1(t) \\
\left( \frac{1}{2\rho} + \frac{1}{2\delta} \right) e^{\delta(t-T)} - \frac{1}{2\delta} = \psi_1(t)
\]

\[
\psi_1(T) e^{(\delta + \rho)(t-T)} = \psi_1(t)
\]

\[
\left(1 + \frac{\rho}{\delta}\right) e^{\delta(t-T)} - \frac{\rho}{\delta} = \psi_1(t)
\]

**APPENDIX 12**

From

\[
\frac{1}{2\sqrt{C}} = \psi_1
\]

we have

\[
C = \left( \frac{1}{2\psi_1} \right)^2
\]

\[
C = \left( \frac{\sqrt{\frac{\delta}{T} + \frac{1}{2\delta}} e^{\delta(t-T)} - \frac{1}{2\delta}}{2\sqrt{\frac{\delta}{T}} \psi_1(T) e^{\delta(t-T)}} \right)^2
\]

\[
C = C_T \left( \frac{\left( \frac{\delta}{T} + \frac{1}{2\delta} \right) \psi_1(T) e^{\delta(t-T)} - \frac{\rho}{\delta}}{\sqrt{\frac{\delta}{T}}} \right)^2
\]

\[
C = C_T \left( \left( \frac{1 + \frac{\rho}{\delta}}{\delta} \right) e^{\delta(t-T)} - \frac{\rho}{\delta} \right)^2
\]

\[
\dot{C} = C_T \left( -\rho \left( 1 + \frac{\rho}{\delta} \right) e^{-\rho(t-T)} + \left( \delta + \rho \right) \frac{\rho}{\delta} e^{-\delta(t-T)} \right) \left( \frac{1 + \frac{\rho}{\delta}}{\delta} \right) e^{-\rho(t-T)} - \frac{\rho}{\delta} \right)^2
\]

\[
\dot{C} = \rho C_T \left( -\rho \left( 1 + \frac{\rho}{\delta} \right) e^{-\rho(t-T)} + \left( 1 + \frac{\rho}{\delta} \right) e^{-\delta(t-T)} \right) \left( \frac{1 + \frac{\rho}{\delta}}{\delta} \right) e^{-\rho(t-T)} - \frac{\rho}{\delta} \right)^2
\]

\[
\dot{C} = \rho \left( 1 + \frac{\rho}{\delta} \right) C_T \left( -e^{-\rho(t-T)} + e^{-\delta(t-T)} \right) \left( \frac{1 + \frac{\rho}{\delta}}{\delta} \right) e^{-\rho(t-T)} - \frac{\rho}{\delta} \right)^2
\]

\[
\dot{C} = \rho \left( 1 + \frac{\rho}{\delta} \right) e^{-2\rho(t-T)} C_T \left( -1 + e^{-\delta(t-T)} \right) \left( 1 + \frac{\rho}{\delta} \left( 1 - e^{-\delta(t-T)} \right) \right)
\]

\[
\dot{C} < 0
\]

since

\[
1 + \frac{\rho}{\delta} \left( 1 - e^{-\delta(t-T)} \right) > 0
\]

**APPENDIX 13**
We have
\[ \dot{R} = R \left( \frac{1}{\sqrt{x}} - \frac{\dot{x}}{x} \right) - C \frac{x}{x} \] (86)
but now
\[ \delta = -\frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} \frac{\dot{x}}{x} \] (87)
so
\[ \dot{R} = -2\delta R - C \frac{x}{x} < 0 \] (88)
\[ \dot{R} = 2\delta R - \frac{C_T \left( \left( \sqrt{x_T} + \frac{1}{2\rho} \right) e^{\delta(t-T)} - \frac{1}{2\rho} \right) e^{-(\delta + \rho)(t-T)} \right)^2}{\left( \left( \sqrt{x_T} + \frac{1}{2\rho} \right) e^{\delta(t-T)} - \frac{1}{2\rho} \right)^2} \] (89)
with
\[ x(t) \rightarrow x_T \equiv \frac{x}{R_{ss}} = \left( \frac{1}{2\rho} \right)^2 \] (90)
\[ C(t) \rightarrow C_T = \omega \hat{S} \begin{bmatrix} \alpha \\ \rho \end{bmatrix} \] (91)
\[ \dot{R} = -2\delta R - \frac{\omega \hat{S} e^{-2(\delta + \rho)(t-T)}}{\left( \frac{1}{2\rho} \right)^2} \] (92)
\[ R(t) = \omega \hat{S} \exp(-2\delta t + 2\delta T - 2\rho t + 2\rho T) + e^{-2\delta t} D, \] (93)
\[ R(T) = \omega \hat{S} + e^{-2\delta T} D = \omega \hat{S} \]
\[ D = 0 \]
so
\[ R(t) = \omega \hat{S} e^{-2(\delta + \rho)(t-T)} \]
\[ \frac{\dot{R}(t)}{R} = -2(\delta + \rho) < 0 \]

**APPENDIX 14**

\[ K = xR \]
so
\[ \delta + \frac{1}{2} x^{-\frac{3}{2}} = + \frac{1}{2} \dot{x} \] (95)

\[ \frac{\dot{K}}{K} = \frac{\dot{x}}{x} + \frac{\dot{R}}{R} = \frac{\dot{x}}{x} - 2(\delta + \rho) \]

\[ \frac{\dot{K}}{K} = \frac{1}{\sqrt{x}} - 2\rho \]

Since \( \frac{\dot{x}}{x} > 0 \) then \( \frac{d}{dt} \left( \frac{1}{\sqrt{x}} \right) < 0 \) so \( \frac{1}{\sqrt{x(t)}} > \frac{1}{\sqrt{x(T)}} = 2\rho \) for all \( t < T \) and therefore
\[ \frac{\dot{K}}{K} > 0. \]

**APPENDIX 15**

Moreover substituting \( x \) and \( R \) yields

\[ K(t) = x(t) R(t) \]

\[ K(t) = \left( \left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{\delta(t-T)} - \frac{1}{2\delta} \right)^2 \omega \hat{S} e^{-2(\delta+\rho)(t-T)} \]

there exist a smooth path reaching \( K_{ss} \) at \( T \) is \( K_0 \) satisfies

\[ K_0 = K(0) = \omega \hat{S} \left( \left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{-\delta T} - \frac{1}{2\delta} \right) e^{(\delta+\rho)T} \]

Let \( g(T) = \left( \left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{-\delta T} - \frac{1}{2\delta} \right) e^{(\delta+\rho)T} \)

\[ g(T) = \left( \sqrt{x_T} + \frac{1}{2\delta} \right) e^{\delta T} - \frac{1}{2\delta} e^{(\delta+\rho)T} \]

\[ g'(T) = \rho \left( \frac{1}{2\rho} + \frac{1}{2\delta} \right) e^{\delta T} - (\delta + \rho) \frac{1}{2\delta} e^{(\delta+\rho)T} \]

\[ g'(T) = \frac{1}{2\delta} e^{\delta T} \left( \rho \left( \frac{\delta}{\rho} + 1 \right) - (\delta + \rho) e^{\delta T} \right) \]

\[ g'(T) = \frac{1}{2\delta} e^{\delta T} (\delta + \rho) (1 - e^{\delta T}) < 0 \]

\[ \frac{dK_0}{dT} = \omega \hat{S} g(T) g'(T) < 0 \]
References


