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Markov-Perfect Nash Equilibria in a Class of Resource Games*

Gerhard Sorger†

Résumé / Abstract

On analyse un modèle standard de l'exploitation des ressources renouvelables par des agents non-coopératifs. Dans le cas où les ressources sont suffisamment productives, on démontre l'existence d'un continuum d'équilibres Markov-parfaits de Nasch (EMPN). Quoique ces équilibres entrainent la surconsommation des ressources, on peut prouver que pour chaque T > 0, il y a des EMPN ayant la propriété que le stock de ressources demeure dans un voisinage arbitrairement petit de l'état stationnaire optimal pendant au moins T périodes. De plus, on obtient une condition nécessaire et suffisante pour que l'exploitation maximale des ressources soit un EMPN. On démontre que cette condition est vérifiée dans le cas où soit il y a beaucoup d'agents, soit les agents sont impatients, soit la capacité de chaque agent est grande.

A standard model of the exploitation of a renewable resource by non-cooperating agents is considered. Under the assumption that the resource is sufficiently productive we prove that there exist infinitely many Markov-perfect Nash equilibria (MPNE). Although these equilibria lead to overexploitation of the resource (tragedy of the commons) it is shown that for any T > 0 there exist MPNE with the property that the resource stock stays in an arbitrary small neighborhood of the efficient steady state for at least T time periods. Furthermore, we derive a necessary and sufficient condition for maximal exploitation of the resource to qualify as a MPNE and show that this condition is satisfied if there are sufficiently many players, or if the players are sufficiently impatient, or if the capacity of each player is sufficiently high.

Mots Clés : Ressources renouvelables, jeu différentiel, équilibres Markov-parfaits, équilibres multiples

Keywords : Renewable resources, differential game, Markov-perfect equilibria, multiple equilibria

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1 Introduction

Dynamic games are a very useful analytical tool for the theory of resource allocation and capital accumulation under imperfect competition. Over the last two decades a number of authors have studied the fundamental issues of existence, uniqueness, and efficiency of non-cooperative equilibria in such games (see, e.g., [1, 3, 4, 5, 6, 7, 8, 10, 11, 13]). The standard model underlying these papers involves a single resource stock (or capital stock) which can be used by finitely many agents. The papers differ from each other mainly in the way how time is modelled (as a discrete variable or a continuous one) and in the assumptions imposed on the utility functions and the growth function of the resource, respectively.

In the present paper we are concerned with a very general form of this common property resource game with n identical players and a continuous time variable. The growth function of the natural resource and the utility functions of the players are assumed to satisfy standard concavity and smoothness assumptions. In addition, we assume that the elasticity of intertemporal substitution of the utility function is bounded below by $n/(n-1)$ and that there exists an upper limit for the resource extraction rate of each agent. Our model is therefore a generalization of the one in [5] where the elasticity of intertemporal substitution was assumed to be constant and equal to $n/(n-1)$.

For the first main result we assume that the resource is sufficiently productive and prove that this implies that there exist infinitely many symmetric Nash equilibria of the game. They consist of stationary Markovian strategies which means that the actions of the players depend only on the present state of the game (the resource stock) and not on past states, the actions of their opponents, or time. The proof is based on existence theorems for solutions to ordinary differential equations. More specifically, we derive an auxiliary differential equation from the Hamilton-Jacobi-Bellman equation which has to be satisfied by the policy functions of the players. This equation is shown to have a solution from which a solution to the original Hamilton-Jacobi-Bellman equation can be constructed. The fact that there exist infinitely many symmetric Markov-perfect Nash equilibria was already discussed in the more restricted model of [5] were the Hamilton-Jacobi-Bellman equation could be solved explicitly. It shows that neither the symmetry assumption nor the requirement of subgame perfectness is sufficient to reduce the set of Nash equilibria to a finite set.

Efficiency of Markov-perfect Nash equilibria is an issue which has received considerable attention in the literature. Intuition suggests that the lack of cooperation (Nash equilibrium) and retaliation (Markovian strategies) would lead to overexploitation of the resource. This intuition has indeed been confirmed in many studies (e. g., [1, 5, 10, 11]) and the equilibria discussed above do also have the feature of overexploitation. Dutta and Sundaram, on the other hand, have shown in [7, 8] that, in general, underexploitation of the resource cannot be ruled out. In the present paper we demonstrate a weaker but related result. We show that for all $T > 0$ and all $\epsilon > 0$ one can find a Markov-perfect Nash equilibrium such that the equilibrium state trajectory spends at least T time periods in the ϵ-neighborhood of the efficient steady state provided that the initial stock of the resource is sufficiently high. Because T and ϵ can be chosen arbitrarily this can be regarded as an approximative efficiency theorem.

In the second main result of the paper we characterize the conditions under which driving the resource stock down to zero as fast as possible is a Markov-perfect equilibrium. We derive a necessary and sufficient condition for such a scenario to be possible. It is furthermore shown that this condition is satisfied provided that at least one of the following three properties holds: there are sufficiently many players, the players are sufficiently impatient, or the upper limit for the extraction rates is sufficiently high. It is demonstrated that there may exist parameter specifications under which both most rapid extinction and a positive steady state resource stock can coexist as the outcomes of Markov-perfect Nash equilibria.

The model formulation and the assumptions are presented in Section 2 where we also state the two main theorems of the paper. The proofs of these theorems can be found in Section 3. Section 4 presents concluding remarks and open questions. Some more technical results needed in the paper are derived in an appendix.

2 Model formulation and results

We consider a continuous time model of a renewable resource which is simultaneously exploited by n non-cooperating agents. The stock of the resource at time $t \in [0, \infty)$ is denoted by $x(t)$ and the harvesting rate of agent $i \in \{1, 2, \ldots, n\}$ at time t is $c_i(t)$. The natural growth rate of the resource depends on the existing stock $x(t)$ and is given by the function $F(x(t))$. We shall also refer to $x(t)$ as the state of the system. It follows from the above assumptions that the state trajectory $x(\cdot)$ is a solution of the initial value problem

$$\dot{x}(t) = F(x(t)) - \sum_{i=1}^{n} c_i(t)\,,\ x(0) = z \tag{1}$$

where the constant z is the initial stock at time zero.

All n players are identical and maximize the present value of utility derived from consumption. Denoting by $r > 0$ the common discount rate of the agents and by U their common utility function, the objective functional of player $i \in \{1, 2, \ldots, n\}$ can be written as

$$\int_0^\infty e^{-rt} U(c_i(t))\, dt. \tag{2}$$

Each player $i \in \{1, 2, \ldots, n\}$ maximizes her objective functional subject to the feasibility constraint

$$c_i(t) \in C(x(t)) := \begin{cases} \{0\} & \text{if } x(t) = 0, \\ [0, k] & \text{if } x(t) > 0. \end{cases} \tag{3}$$

The interpretation of this constraint is that negative harvesting rates are excluded[1], that nothing can be harvested if the stock size is equal to zero, and that there is a fixed upper bound, $k > 0$, on the feasible harvesting rates due to capacity limitations.

This completes the formulation of the model. The fundamentals of the game are n, F, U, r, k, and z. We are now going to state and discuss the assumptions which will be used throughout the paper.

[1]Examples of negative harvesting are, e.g., breeding fish in a fishery model or reforestation in a tree cutting model.

A1: The growth function $F : [0,\infty) \mapsto I\!R$ is twice continuously differentiable, strictly concave, and satisfies $F(0) = F(1) = 0$. The initial stock z lies in the interval $[0,1]$.

Assumption A1 is often made in renewable resource models (see, e.g., [2]) and it states that positive growth is only possible as long as the stock size $x(t)$ is smaller than some given constant which, by suitable normalization, is chosen to be equal to one.[2] It follows from this assumption that F is strictly positive on the interval $(0,1)$ and strictly negative on $(1,\infty)$. Because the initial stock z is in $[0,1]$ we know that feasible solutions of the state equation (1) satisfy $x(t) \in [0,1]$ for all $t \in [0,\infty)$. We shall therefore call $[0,1]$ the state space of the model and shall not consider any stock sizes other than those in this interval. Another consequence of A1 is that there exists a unique level $x_0 \in (0,1)$ at which the natural growth rate is maximized. Of course, x_0 is determined by the equation $F'(x_0) = 0$ and it holds that $F'(x) > 0$ for all $x \in [0,x_0)$ and $F'(x) < 0$ for all $x \in (x_0,1]$. For later reference let us also define the stock levels x_1 and x_n by $x_1 = \inf\{x \,|\, F'(x) \leq r, x \in [0,1]\}$ and $x_n = \inf\{x \,|\, F'(x) \leq nr, x \in [0,1]\}$. Note that A1 implies that $x_n \leq x_1 < x_0$.

A2: The utility function $U : [0,k] \mapsto I\!R$ is continuous, twice continuously differentiable on $(0,k]$, and strictly concave. It holds that $U'(c) > 0$ for all $c \in (0,k]$ and $\lim_{c \searrow 0} U'(c) = \infty$.[3]

Assumption A2 is a standard assumption and need not be commented on. It is well known (see, e.g., [2]) that under A1 and A2 there exists a unique solution to the problem of maximizing the sum of the utility functions of all agents subject to the constraints (1) and (3). This is called the efficient solution. If the initial stock z is strictly positive, then the stock size $x(t)$ in the efficient solution converges to the steady state x_1 as t approaches infinity. We shall therefore call x_1 the efficient steady state.

A3: It holds that $k > F(x_0)/n$.

[2]The stock size beyond which growth becomes negative can be interpreted as the carrying capacity of the environment.

[3]For any function f we denote by $\lim_{x \searrow y} f(x)$ the limit of $f(x)$ as x approaches y from above. Similarly, we will write $\lim_{x \nearrow y} f(x)$ for the limit of $f(x)$ as x approaches y from below.

This assumption requires that the harvesting capacity of all agents together, nk, exceeds the maximal natural growth rate $F(x_0)$. Therefore, it is in principle possible for the n agents to completely exhaust the resource from any initial stock $z \in [0, 1]$.

A4: The function $\beta : (0, k] \mapsto I\!R$ defined by $\beta(c) = -cU''(c)/U'(c)$ has a locally Lipschitz continuous extension to the interval $[0, k]$ and $\beta_0 := \lim_{c \searrow 0} \beta(c) > 0$.

A5: For all $c \in (0, k]$ it holds that $\beta(c) \leq (n-1)/n$.

Note that $\beta(c)^{-1}$ is the elasticity of intertemporal substitution at the consumption level c. Assumption A4 requires that this elasticity is a sufficiently smooth function of c and remains bounded at $c = 0$. Assumption A5, on the other hand, relates the elasticity of intertemporal substitution to the number of players by requiring that the former is sufficiently high at all consumption levels.

The analysis in [5] makes also use of properties A1 and A3 but it uses stronger assumptions concerning the utility function U. More specifically, instead of A2, A4, and A5 it is assumed in [5] that $\beta(c) = (n-1)/n$ for all $c \in (0, k]$, i. e., that the utility function exhibits constant elasticity of intertemporal substitution.[4]

A strategy σ for player $i \in \{1, 2, \ldots, n\}$ is any rule that determines this player's consumption, $c_i(t)$, at each time $t \in [0, \infty)$ as a function of t, the realized states $x(\tau)$ for $\tau \in [0, t]$, and the consumption paths of all players previous to time t. We call σ a stationary Markovian strategy if it determines $c_i(t)$ as a function of the current state $x(t)$ only, that is, if there exists a function $\phi : [0, 1] \mapsto [0, k]$ such that $c_i(t) = \phi(x(t))$ holds for all $t \geq 0$. In this case, the function ϕ is called the policy function of player i.

It is well known that for a differential game to be well defined one has to restrict the set of strategies that are available to the players.[5] One possibility is to allow only stationary Markovian strategies with Lipschitz continuous

[4]Only the case of two players, $n = 2$, is explicitly treated in [5] but the results can easily be generalized to $n > 2$ if one uses the utility function $U(c) = c^{1/n}$ instead of the square root function.

[5]See for example the discussion in [9, Sec. 13.3.4] as well as the references listed there.

policy functions. Such a restriction, however, is very problematic as there is hardly any economic justification for it. Here we shall use a more general approach and to this end we introduce the following definition.

Definition 1 Let $G = (n, F, U, r, k, z)$ be a given game and, for each player $i \in \{1, 2, \ldots, n\}$, let σ_i be a strategy. The n-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is said to be admissible for G if the following conditions are satisfied for all $i \in \{1, 2, \ldots, n\}$:
(a) $c_i(t)$ is well defined for all $t \in [0, \infty)$,
(b) the function $t \mapsto c_i(t)$ is measurable,
(c) $c_i(t) \in [0, k]$ for all $t \in [0, \infty)$ and $c_i(t) = 0$ whenever $x(t) = 0$, and
(d) the initial value problem (1) has a unique absolutely continuous solution.

The conditions in this definition are the minimal requirements for the state trajectory in (1) and the objective functionals in (2) to be well defined and unique. On the other hand, we do not exclude general (history dependent) strategies like trigger strategies from consideration. We shall only consider admissible n-tuples of strategies throughout the paper.

Definition 2 Let $G = (n, F, U, r, k, z)$ be a given game and $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ an n-tuple of strategies that is admissible for G. Furthermore, denote by $J_i(\sigma_1, \sigma_2, \ldots, \sigma_n)$ the value of player i's objective functional in (2) when the players use the strategies σ_1, σ_2, $\ldots$, and σ_n, respectively. We say that $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a Nash equilibrium if for all $i \in \{1, 2, \ldots, n\}$ and for any strategy $\tilde{\sigma}_i$ such that $(\sigma_1, \sigma_2, \ldots, \tilde{\sigma}_i, \ldots, \sigma_n)$ is admissible for G it holds that $J_i(\sigma_1, \sigma_2, \ldots, \tilde{\sigma}_i, \ldots, \sigma_n) \leq J_i(\sigma_1, \sigma_2, \ldots, \sigma_i, \ldots, \sigma_n)$.

A Nash equilibrium consisting of stationary Markovian strategies will be called a stationary Markovian Nash equilibrium. If $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a Nash equilibrium not only for $G = (n, F, U, r, k, z)$ but also for all games of the form (n, F, U, r, k, x) where $x \in [0, 1]$ then we call $\underline{\sigma}$ subgame-perfect. A stationary Markovian Nash equilibrium which is subgame-perfect is also called a Markov-perfect Nash equilibrium. Since subgame-perfect equilibria are independent of the initial state we may omit the initial state from the specification of the game and simply say that $\underline{\sigma}$ is a subgame-perfect Nash equilibrium of the game $G = (n, F, U, r, k)$. If a Nash equilibrium is such that all players use the same strategy then we say that the equilibrium is symmetric. Because in the game under consideration all players are identical it is natural to focus on symmetric equilibria. We are now ready to state the first result of the paper.

Theorem 1 *Let $G = (n, F, U, r, k)$ be a game satisfying assumptions A1 - A5 as well as the condition*

$$F(x_1) > nrx_1 > 0. \tag{4}$$

(a) *There exists a continuum of different Markov-perfect Nash equilibria for G. These equilibria are symmetric and each player's strategy is defined by a policy function $\phi_\alpha : [0,1] \mapsto [0,k]$ where α is a parameter which can be chosen from a non-empty real interval (α_0, α_1).*
(b) *The policy functions $\phi_\alpha(x)$, $\alpha \in (\alpha_0, \alpha_1)$, are continuous and strictly increasing with respect to x on the interval $[0, x_1]$ and they satisfy $\phi_\alpha(0) = 0$. On the interval $(x_1, 1]$ they are constant and equal to the capacity limit, i. e., $\phi_\alpha(x) = k$ for all $x \in (x_1, 1]$. At $x = x_1$ the policy functions exhibit a jump discontinuity.*
(c) *Let $x_\alpha(t; z)$ denote the state at time t generated by the equilibrium with parameter $\alpha \in (\alpha_0, \alpha_1)$ when the initial state is equal to z. For all $z \in (0, 1]$ and all $\alpha \in (\alpha_0, \alpha_1)$ it holds that $\lim_{t \nearrow \infty} x_\alpha(t; z) = \bar{x}_\alpha$ where $\bar{x}_\alpha \in (0, x_n)$.*
(d) *For all initial states $z \geq x_1$, all $\epsilon > 0$, and all $T > 0$ there exists a parameter $\alpha \in (\alpha_0, \alpha_1)$ such that $\lambda\left(\{t \in [0, \infty) \,|\, |x_\alpha(t; z) - x_1| < \epsilon\}\right) > T$.*[6]

Part (a) of the theorem deals with the existence of infinitely many Markov-perfect Nash equilibria, part (b) describes the shape of the policy functions, and part (c) shows that the long-run steady states of these equilibria, $\bar{x}_\alpha$, are strictly smaller than x_n and therefore also strictly smaller than the efficient steady state x_1. In the usual terminology (see, e. g., [7]) the result from (c) says that the equilibria constructed in the theorem lead to a tragedy of the commons. Part (d), on the other hand, shows that one can find Markov-perfect Nash equilibria for which the resource stock stays in an arbitrary small neighborhood of the efficient steady state for arbitrary long time provided the initial stock z is sufficiently high. One can therefore approximate the efficient steady state by state trajectories generated from Markov-perfect Nash equilibria.

Condition (4) is sufficient but not necessary for the results (a) - (d) to hold. It is easy to prove that (4) implies $x_n > 0$ which, according to the definition of x_n, is equivalent to the condition $F'(0) > rn$.[7] Therefore, (4) can be

[6] By λ we denote the Lebesgue measure on the real line.

[7] As a matter of fact, $x_n > 0$ follows immediately from part (c) of the theorem but it can also be proved directly using only A1 and (4).

interpreted by saying that the slope of the function $F(x)$ on the interval $[0, x_1)$ has to be sufficiently high as compared to the discount rate r and the number of players n. In other words, the resource has to be sufficiently productive for small stock sizes.

In the case where the resource is not very productive as compared to r and n intuition suggests that the situation where all players exhaust the resource at a maximal rate qualifies as an equilibrium. This makes sense because if r is large then the players are very impatient and they do not care much about the conservation of the resource, and if n is large then the fierce competition reinforces the tragedy of the commons. In the following result we confirm this intuitive reasoning by deriving a necessary and sufficient condition for maximal exploitation to be an equilibrium. We also show that a high capacity limit k implies that maximal exploitation is an equilibrium.

Theorem 2 **(a)** *If $G = (n, F, U, r, k)$ is a game satisfying A1 - A5 then the following two conditions are equivalent:*

1.

$$U'(k) \geq \frac{U(k) - U(0)}{nk - F(x_1)} \exp\left[-r \int_0^{x_1} \frac{dy}{nk - F(y)}\right] \tag{5}$$

2. The n-tuple $(\sigma, \sigma, \ldots, \sigma)$ is a Markov-perfect Nash equilibrium, where the strategy σ is defined by the policy function

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0, \\ k & \text{if } x \in (0, 1]. \end{cases} \tag{6}$$

(b) *If $G = (n, F, U, r, k)$ is a game satisfying A1 - A5 and $r \geq F'(0)$ then condition (5) holds.*
(c) *Assume F, U, r, and k are given such that A1, A2, and A4 are satisfied and such that $\sup\{\beta(c) \mid c \in (0, k]\} < 1$. If n is sufficiently large then A3, A5 and (5) are also satisfied.*
(d) *Assume n, F, U, and r are given such that A1, A2, A4, and A5 are satisfied and such that $\lim_{c \to \infty} \beta(c) < (n-1)/n$. If k is sufficiently large then A3 and (5) are also satisfied.*

Part (a) of the theorem states a necessary and sufficient condition for maximal exploitation to be an equilibrium. This condition is quite easy to check for any given game (n, F, U, r, k). Parts (b), (c), and (d), on the other hand, show that any of the following properties is, ceteris paribus, a sufficient condition for maximal exploitation to be an equilibrium: the discount rate r is high, the number of players n is large, or the capacity limit k is high.

It should be noted that the conditions of Theorems 1 and 2 do not exclude each other. There may be situations in which both (4) and (5) are satisfied so that there exist both the infinitely many equilibria described in Theorem 1 as well as the most rapid extinction equilibrium of Theorem 2. For example assume that n, F, U, and r are given such that A1, A2, A4, A5, (4), and $\lim_{c\to\infty} \beta(c) < (n-1)/n$ are satisfied. Since (4) does not depend on the value of k, Theorem 2(d) tells us that we may increase k without affecting the validity of any of A1, A2, A4, A5, and (4), and at the same time ensure that A3 and (5) hold as well.

For a more specific example consider the case where $F(x) = mx(1-x)$, $U(c) = \sqrt{c}$, $r = 0.1$, and $k = 0.5$. It is straightforward to check that in this case assumptions A1 - A5 are satisfied whenever n and m are such that $n \geq 2$ and $2n > m > 0$. If $n = 3$ and $m = 1$ then assumptions A1 - A5 as well as conditions (4) and (5) hold true. Therefore, our theorems imply that most rapid extinction is a Markov-perfect Nash equilibrium but that there are also infinitely many other Markov-perfect Nash equilibria with strictly positive steady state resource stocks. If, on the other hand, $n = 2$ and $m = 0.2$, then A1 - A5 still hold but neither (4) nor (5) is satisfied so that all we can infer from the results of this paper is that most rapid extinction is not an equilibrium. In this particular case we have $\beta(c) = (n-1)/n$ for all $c \in (0, k]$ so that a result from [5] guarantees the existence of Markov-perfect equilibria. However, a slight perturbation of the utility function (for example $U(c) = c^{51/101}$ instead of $U(c) = \sqrt{c}$) would lead to a model in which neither the results of the present paper nor those of [5] ensure the existence of Markov-perfect Nash equilibria.

3 Proofs

3.1 Proof of Theorem 1: The result will be established in a series of steps. The general idea is to construct functions $\phi_\alpha : [0, 1] \mapsto [0, k]$ and to prove that if all agents except for agent i choose the strategy defined by the policy function ϕ_α then player i's optimal strategy can also be described by ϕ_α. So let us assume that $n-1$ players have chosen the policy function ϕ_α and consider the utility maximization problem of the remaining agent. The Hamilton-Jacobi-Bellman equation for this optimal control problem is

$$rV_\alpha(x) = \max\left\{U(c) + V_\alpha'(x)\left[F(x) - c - (n-1)\phi_\alpha(x)\right] \,\middle|\, c \in C(x)\right\}$$

where $V_\alpha : [0,1] \mapsto I\!R$ denotes the optimal value function. The main step of the proof is to find piecewise continuously differentiable functions $V_\alpha : [0,1] \mapsto I\!R$ which solve this equation in the sense that for all $x \in [0,1]$ where $V_\alpha'(x)$ is defined, the conditions

$$\begin{aligned} \phi_\alpha(x) &= \operatorname{argmax}\left\{U(c) + V_\alpha'(x)\left[F(x) - c - (n-1)\phi_\alpha(x)\right] \,\middle|\, c \in C(x)\right\} && (7)\\ rV_\alpha(x) &= U(\phi_\alpha(x)) + V_\alpha'(x)\left[F(x) - n\phi_\alpha(x)\right] && (8)\end{aligned}$$

are satisfied. The remaining steps are then to show that the n-tuple $(\phi_\alpha, \phi_\alpha, \ldots, \phi_\alpha)$ is admissible, that equations (7) and (8) do indeed imply the optimality of ϕ_α, and that assertions (c) and (d) of the theorem hold true.

As can be seen from Theorem 1(b), $\phi_\alpha(x)$ is a boundary solution of the maximization problem in (7) for $x \in (x_1, 1]$. For $x \in (0, x_1]$, on the other hand, it is an interior solution and for $x = 0$ we must have $\phi_\alpha(x) = 0$ by (3). We shall first deal with the case where the capacity constraint is not binding and then with the easier case of a boundary solution.

3.2 Construction of $\phi_\alpha(x)$ and $V_\alpha(x)$ for $x \in [0, x_1]$: Because of (3) every feasible policy function ϕ_α must satisfy

$$\phi_\alpha(0) = 0. \tag{9}$$

Since the growth rate at zero satisfies $F(0) = 0$ it follows that the only feasible solution of the state equation (1) with an initial state $z = 0$ is $x(t) = 0$ for all $t \in [0, \infty)$. The maximal utility that each player can attain in this case is therefore given by

$$V_\alpha(0) = \int_0^\infty e^{-rt} U(0)\, dt = U(0)/r. \tag{10}$$

Now let us consider the case $x \in (0, x_1]$. If $\phi_\alpha(x)$ is an interior solution in (7) then it must satisfy the first order condition

$$U'(\phi_\alpha(x)) = V_\alpha'(x). \tag{11}$$

We can substitute this into (8) to obtain $rV_\alpha(x) = U(\phi_\alpha(x)) + U'(\phi_\alpha(x))\,[F(x) - n\phi_\alpha(x)]$. Assuming differentiability of ϕ_α and V_α we can differentiate this equation with respect to x.[8] Together with (11) this yields

$$rU'(\phi_\alpha(x)) = U'(\phi_\alpha(x))\phi'_\alpha(x) + U''(\phi_\alpha(x))\phi'_\alpha(x)\,[F(x) - n\phi_\alpha(x)] + U'(\phi_\alpha(x))\,[F'(x) - n\phi'_\alpha(x)]\,.$$

Dividing by $U'(\phi_\alpha(x))$ (which is strictly positive by A2) and using the definition of β from A4 we can rewrite this equation as follows.

$$\phi'_\alpha(x) = G(x, \phi_\alpha(x)) := \frac{F'(x) - r}{n - 1 - n\beta(\phi_\alpha(x)) + \beta(\phi_\alpha(x))F(x)/\phi_\alpha(x)}\,. \tag{12}$$

Now consider the domain $D = \{(x,\phi)\,|\,0 < x \leq x_1, 0 < \phi < k\}$ and the curve $, = \{(x,\phi)\,|\,0 < x \leq x_1, \phi = F(x)/n\}$.[9] First note that because of A3 the curve , is contained in D, that is, the upper boundary of D lies strictly above , . Furthermore, assumptions A1 - A5 and the definition of x_1 imply that $G(x,\phi) \geq 0$ for all $(x,\phi) \in D$ with the strict inequality holding whenever (x,ϕ) is in the interior of D. This means that the vector field defined by equation (12) is pointing upwards everywhere on D. Now consider any point $(x_1, \alpha) \in D$ with $\alpha \in [F(x_1)/n, k)$, that is, any point on the right boundary of D which lies above , .[10] Our first lemma states that for any such point there exists a unique trajectory of equation (12) which is defined over the interval $x \in (0, x_1]$ and which terminates at that point.

Lemma 1 *For all $\alpha \in [F(x_1)/n, k)$ there exists a unique solution $\phi_\alpha : (0, x_1] \mapsto I\!\!R$ of the differential equation (12) which satisfies $\phi_\alpha(x_1) = \alpha$. Any such trajectory satisfies $0 < \phi_\alpha(x) < k$ for all $x \in (0, x_1]$. It holds that $\phi_\alpha(x)$ is strictly increasing with respect to both x and α.*

PROOF. There exist local solutions to (12) by the Cauchy-Peano existence theorem. Because the function G is locally Lipschitz continuous on D these solutions have to be unique. Now consider any point $P = (x_1, \alpha)$ as described in the lemma (see also Figure 1). Since the vector field defined by

[8] We shall later verify that ϕ_α and V_α are indeed differentiable for $x \in (0, x_1)$ so that this heuristic argument can be made rigoros.

[9] For the following analysis the reader may find it useful to look at Figure 1 where D and , are depicted.

[10] In Figure 1 one such point is indicated and labelled as P. Note that because of A3 we have $k > F(x_1)/n$ so that a continuum of such points exists.

G is pointing upwards on D, we know that any solution terminating in P must be strictly increasing on the interval $(0, x_1]$. Going backwards from P the trajectory therefore cannot leave the domain D through its upper boundary. Similarly, it cannot leave D through its lower boundary, because $\lim_{\phi \searrow 0} G(x, \phi) = 0$ so that the vector field becomes horizontal at the lower boundary. It follows that any local solution can be continued on the entire interval $(0, x_1]$. The other assertions of the lemma are obvious. □

The trajectories ϕ_α mentioned in Lemma 1 terminate at points above the curve , when $x = x_1$. In the following lemma we show that all solution curves ϕ_α with α close to $F(x_1)/n$ cross the curve , exactly once. We also derive an inequality which describes the behavior of these trajectories close to $x = 0$. To simplify the notation we define $\alpha_0 = F(x_1)/n$.

Lemma 2 *Let $\phi_\alpha : (0, x_1] \mapsto [0, k]$ be the trajectories mentioned in Lemma 1. There exists $\bar{\alpha} \in (\alpha_0, k)$ such that for all $\alpha \in [\alpha_0, \bar{\alpha})$ the following is true.*
(a) *There exists $\bar{x}_\alpha \in (0, x_n)$ such that $\phi_\alpha(x) < F(x)/n$ if and only if $x \in (0, \bar{x}_\alpha)$ and $\phi_\alpha(x) > F(x)/n$ if and only if $x \in (\bar{x}_\alpha, x_1]$.*
(b) *There exists $M_\alpha > 0$ such that for all sufficiently small $x > 0$ it holds that $\phi_\alpha(x) \geq M_\alpha x^\gamma$ where $\gamma = [F'(0) - r]/[\beta_0 F'(0)]$.*[11]

Proof. (a) Let us denote by D_- the part of D which lies below , . We start by proving the claim for $\alpha = \alpha_0$. To this end assume by way of contradiction that ϕ_{α_0} does not enter D_- at all. Then we must have $\phi_{\alpha_0}(x) \geq F(x)/n$ for all $x \in (0, x_1]$. Using (12) this yields $\phi'_{\alpha_0}(x) \geq [F'(x) - r]/(n-1)$ for all $x \in (0, x_1]$. Together with $\phi_{\alpha_0}(x) + \int_x^{x_1} \phi'_{\alpha_0}(y)\, dy = \phi_{\alpha_0}(x_1) = \alpha_0 = F(x_1)/n$ and $\phi_{\alpha_0}(x) > 0$ this implies

$$F(x_1)/n > \int_x^{x_1} \frac{F'(y) - r}{n-1}\, dy = \frac{F(x_1) - r x_1}{n-1} - \frac{F(x) - rx}{n-1}.$$

In the limit as x approaches zero this yields $F(x_1) \leq r n x_1$ which is a contradiction to (4). Thus, we have proved that the trajectory ϕ_{α_0} must cross , . By continuity of solutions of (12) with respect to the boundary condition α this implies that also all trajectories ϕ_α with $\alpha \in [\alpha_0, \bar{\alpha})$ must cross , provided that $\bar{\alpha}$ is greater than but sufficiently close to α_0.

To complete the proof of part (a) we have to show that any trajectory ϕ_α with $\alpha \in [\alpha_0, \bar{\alpha})$ can cross the line , only once and that the crossing point $\bar{x}_\alpha$ lies

[11]See A4 for the definition of β_0.

in $(0, x_n)$. To this end consider the direction of the vector field defined by (12) along the curve , . We have $G(x,\phi)|_\Gamma = G(x, F(x)/n) = [F'(x) - r]/(n-1)$. Since the slope of , is given by $F'(x)/n$ it follows that the vector field along , points out of D_- if $[F'(x) - r]/(n-1) > F'(x)/n$ and it points into D_- if $[F'(x) - r]/(n-1) < F'(x)/n$. A simple calculation shows that the former is the case if and only if $F'(x) > nr$ and therefore $x < x_n$ whereas the latter inequality is equivalent to $F'(x) < nr$, that is, $x > x_n$.[12] Since $\phi_\alpha(x)$ is above , for $x = x_1$ the last crossing point $\bar{x}_\alpha$ must satisfy $\bar{x}_\alpha < x_n$ (the trajectory is leaving D_- so that the vector field must point outwards!). But then there cannot be another crossing to the left of $\bar{x}_\alpha$ so that $\bar{x}_\alpha$ must be the only crossing point. This proves part (a) of the lemma.

(b) First note that $\lim_{x \searrow 0} \phi_\alpha(x) = 0$ must hold because we have $0 < \phi_\alpha(x) < F(x)/n$ for all $x \in (0, \bar{x}_\alpha)$ and $\lim_{x\searrow 0} F(x) = 0$. Using (12) we obtain that

$$\frac{x\phi'_\alpha(x)}{\phi_\alpha(x)} = \frac{x[F'(x) - r]}{[n - 1 - n\beta(\phi_\alpha(x))]\phi_\alpha(x) + \beta(\phi_\alpha(x))F(x)}.$$

Because of $\phi_\alpha(x) > 0$ and A5 this implies

$$\frac{x\phi'_\alpha(x)}{\phi_\alpha(x)} \leq \frac{F'(x) - r}{\beta(\phi_\alpha(x))F(x)/x}. \tag{13}$$

Now note that $\lim_{x\searrow 0} F'(x) = \lim_{x\searrow 0} F(x)/x = F'(0)$ and $\lim_{x\searrow 0} \beta(\phi_\alpha(x)) = \beta_0$ so that the limit as $x \searrow 0$ of the right hand side of (13) is equal to γ. Because the right hand side of (13) is a smooth function of x it follows that that there exists a constant K such that for all sufficiently small x the condition $-x\phi'_\alpha(x)/\phi_\alpha(x) \geq -\gamma + Kx$ holds. From Lemma A in the appendix we conclude that there exists a constant M_α such that $\phi_\alpha(x) \geq M_\alpha x^\gamma$ is satisfied for all sufficiently small x. □

Now that we have defined the functions $\phi_\alpha(x)$ for $x \in [0, x_1]$ we proceed to define $V_\alpha(x)$. Because of (10) and (11) we must have

$$V_\alpha(x) = U(0)/r + \int_0^x U'(\phi_\alpha(y))\, dy \tag{14}$$

for all $x \in [0, x_1]$ and $\alpha \in [\alpha_0, k)$. We have to verify, however, that the integral on the right hand side is well defined because for $y \searrow 0$ we have $\phi_\alpha(y) \searrow 0$ and henceforth $U'(\phi_\alpha(y)) \nearrow \infty$. This is the purpose of the following lemma.

[12] At this point we urge the reader again to consult Figure 1 for the intuition of the following argument.

Lemma 3 *The integral in (14) is finite for all $x \in [0, x_1]$ and all $\alpha \in [\alpha_0, \bar{\alpha})$.*

PROOF. Assumption A4 implies that there exists a constant K such that $\beta(c) \leq \beta_0 + Kc$ for all sufficiently small c. Using Lemma A from the appendix with $f = U'$ and $\delta = \beta_0$ we obtain the existence of a constant M such that

$$U'(c) \leq Mc^{-\beta_0} \tag{15}$$

for all sufficiently small $c > 0$. Since $y \mapsto U'(\phi_\alpha(y))$ is a continuous and strictly positive function of $y \in (0, x_1]$ we only need to rule out the case $\lim_{w \searrow 0} \int_w^x U'(\phi_\alpha(y))\, dy = +\infty$ for some $x > 0$. Because of Lemma 2(b) we know that $\phi_\alpha(y) \geq M_\alpha y^\gamma$ for all $y \in (0, x)$ provided that x is small enough. Using this together with (15) we obtain

$$0 < \int_w^x U'(\phi_\alpha(y))\, dy \leq \int_w^x M M_\alpha^{-\beta_0} y^{-\beta_0 \gamma}\, dy.$$

By the definition of γ and because of $F'(0) > r$ (which follows from (4)) we have $-\beta_0 \gamma = [r - F'(0)]/F'(0) \in (-1, 0)$. This implies that the integral on the right hand side of the above inequality remains bounded as w approaches zero and, henceforth, proves that $V_\alpha(x)$ is well defined by (14) for all $x \in (0, x_1]$. □

Lemma 4 *For all $\alpha \in (\alpha_0, \bar{\alpha})$ the following is true:*
(a) *The functions ϕ_α and V_α defined above are continuous on $[0, x_1]$ and continuously differentiable on $(0, x_1)$.*
(b) *Conditions (7) and (8) hold for all $x \in (0, x_1)$.*

PROOF. Part (a) follows immediately from the construction of the functions and from the properties we have shown before. To prove part (b) first note that the maximand in (7) is a strictly concave function with respect to c. To prove that $c = \phi_\alpha(x)$ is the unique maximizer it is therefore sufficient to verify the first order condition $U'(\phi_\alpha(x)) = V'_\alpha(x)$. Since $V_\alpha(x)$ was defined by (14) this condition is automatically satisfied. To verify (8) we have to show that

$$U(0) + r \int_0^x U'(\phi_\alpha(y))\, dy = U(\phi_\alpha(x)) + U'(\phi_\alpha(x))[F(x) - n\phi_\alpha(x)]. \tag{16}$$

As in the proof of Lemma 3 one can show that $U'(\phi_\alpha(x)) \leq M M_\alpha^{-\beta_0} x^{-\beta_0 \gamma}$ holds for all sufficiently small $x > 0$. Furthermore, we have $F(x) \leq F'(0)x$ for all $x \in [0,1]$ because of concavity of F. Putting these inequalities together and using the fact that $0 \leq \phi_\alpha(x) \leq F(x)/n$ holds for all $x \in [0, \bar{x}_\alpha]$ we obtain

$$0 \leq U'(\phi_\alpha(x))[F(x) - n\phi_\alpha(x)] \leq U'(\phi_\alpha(x))F(x) \leq F'(0)MM_\alpha^{-\beta_0}x^{1-\beta_0\gamma}$$

for all sufficiently small $x > 0$. Since $\beta_0\gamma \in (0,1)$ this implies that $\lim_{x\searrow 0} U'(\phi_\alpha(x))[F(x) - n\phi_\alpha(x)] = 0$. It follows that (16) holds for $x = 0$. To show that (16) also holds true for all $x \in (0, x_1)$ it suffices therefore to prove that the derivative with respect to x of the left hand side of (16) equals the derivative with respect to x of the right hand side of (16) for all $x \in (0, x_1)$. Using (12) it is straightforward to show that this is indeed the case. □

3.3 Construction of $\phi_\alpha(x)$ and $V_\alpha(x)$ for $x \in (x_1, 1]$: The construction of $\phi_\alpha(x)$ and $V_\alpha(x)$ for $x \in (x_1, 1]$ is much easier than the construction for $x \in [0, x_1]$. As has already been mentioned we construct an equilibrium in which the optimal harvesting rate is as high as possible if the stock of the resource exceeds the value x_1. Formally, we have

$$\phi_\alpha(x) = k \tag{17}$$

for all $\alpha \in [\alpha_0, k)$ and all $x \in (x_1, 1]$. Substituting this into (8) and rearranging we obtain

$$V_\alpha'(x) = [rV_\alpha(x) - U(k)]/[F(x) - nk]. \tag{18}$$

Since V_α has to be continuous we can use the value $V_\alpha(x_1)$ from the preceding subsection as an initial value for the above ordinary differential equation. The unique solution of the resulting initial value problem is given by

$$V_\alpha(x) = \frac{U(k)}{r} - \left[\frac{U(k)}{r} - V_\alpha(x_1)\right] \exp\left[-r\int_{x_1}^{x} \frac{dy}{nk - F(y)}\right]. \tag{19}$$

By assumption A3 the integral in this definition is finite so that $V_\alpha(x)$ is well defined. This completes the construction of $\phi_\alpha(x)$ and $V_\alpha(x)$. It remains to verify conditions (7) and (8).

Lemma 5 *There exists $\alpha_1 \in (\alpha_0, \bar{\alpha})$ such that for all $\alpha \in (\alpha_0, \alpha_1)$ the following is true:*
(a) *The functions ϕ_α and V_α defined above are continuously differentiable on $(x_1, 1]$.*
(b) *Conditions (7) and (8) hold for all $x \in (x_1, 1]$.*

PROOF. Part (a) is obvious. To prove part (b) note that the maximand in (7) is a strictly concave function with respect to c. It is therefore sufficient to verify the first order condition. In the present case we have a boundary solution $\phi_\alpha(x) = k$ so that the first order condition reads $U'(k) \geq V'_\alpha(x)$. Because of (19) this is equivalent to

$$U'(k) \geq \frac{U(k) - rV_\alpha(x_1)}{nk - F(x)} \exp\left[-r\int_{x_1}^{x} \frac{dy}{nk - F(y)}\right].$$

It is straightforward to show that the right hand side of this inequality is a strictly decreasing function with respect to x on the interval $(x_1, 1]$. Therefore, the first order condition is satisfied for all $x \in (x_1, 1]$ if and only if it is satisfied for $x = x_1$. For $x = x_1$, however, the above inequality is given by

$$U'(k) \geq \frac{U(k) - rV_\alpha(x_1)}{nk - F(x_1)}. \tag{20}$$

We shall first assume that $\alpha = \alpha_0$ and show that the strict inequality holds in (20). By continuity it follows then that (20) holds for all $\alpha \in [\alpha_0, \alpha_1)$ with α_1 sufficiently close to α_0.

Because of $\phi_{\alpha_0}(x_1) = \alpha_0 = F(x_1)/n$ we obtain from Lemma 4 that $rV_{\alpha_0}(x_1) = U(\alpha_0) + V'_{\alpha_0}(x_1)[F(x_1) - n\alpha_0] = U(\alpha_0)$. Condition (20) can therefore be rewritten as

$$U'(k) > \frac{U(k) - U(\alpha_0)}{n(k - \alpha_0)}.$$

Since $0 < \alpha_0 < k$ it follows from Lemma B in the appendix that the above inequality holds. This completes the proof of (7) for $x \in (x_1, 1]$. Condition (8) holds by construction of V_α. □

3.4 Admissibility, optimality, and dynamics: The following result shows that the n-tuple $(\sigma, \sigma, \ldots, \sigma)$ is admissible for G when σ is the stationary Markovian strategy defined by the policy function ϕ_α.

Lemma 6 *Let $\alpha \in (\alpha_0, \alpha_1)$ be a given parameter and denote by σ the stationary Markovian strategy defined by the policy function ϕ_α. The n-tuple $(\sigma, \sigma, \ldots, \sigma)$ is admissible.*

PROOF. We have seen before that the functions ϕ_α satisfy $\phi_\alpha(0) = 0$ and $0 \leq \phi_\alpha(x) \leq k$ for all $x \in [0, 1]$. Moreover, they are continuously differentiable on the intervals $[0, x_1)$ and $(x_1, 1]$. At $x = x_1$ they have a jump discontinuity. To prove admissibility of $(\sigma, \sigma, \ldots, \sigma)$ it is therefore sufficient to show that the state equation (1) has a well defined and absolutely continuous solution whenever $c_i(t) = \phi_\alpha(x(t))$ holds for all $t \in [0, \infty)$ and all $i \in \{1, 2, \ldots, n\}$. Because of the continuous differentiability of F the only problem can occur when $x(t) = x_1$. Now note that for all $x \in (x_1, 1]$ it holds that $\phi_\alpha(x) = k > F(x)/n$ (see (17) and A3). This implies that the right hand side of (1) is strictly negative whenever $x(t) > x_1$. If x is smaller than x_1 but close to x_1 then we also have $\phi_\alpha(x) > F(x)/n$ because the trajectory ϕ_α constructed in Lemma 1 lies strictly above the curve , for $\alpha > \alpha_0$ and $x \in (\bar{x}_\alpha, x_1]$. Consequently, the right hand side of (1) is strictly negative whenever $x(t)$ is close to x_1 so that existence of absolutely continuous solutions to (1) is guaranteed. □

Lemma 7 *Let σ be the strategy defined by the policy function ϕ_α for some $\alpha \in (\alpha_0, \alpha_1)$. Then it follows that $(\sigma, \sigma, \ldots, \sigma)$ is a symmetric Markov-perfect Nash equilibrium.*

PROOF. One has to show that ϕ_α is an optimal policy function for the optimal control problem of player $i \in \{1, 2, \ldots, n\}$ when all the other players use the policy function ϕ_α. This can be done by a standard argument using the boundedness of the optimal value function V_α and the Hamilton-Jacobi-Bellman equation which was verified in the previous subsections. The fact that the optimal value function V_α is not differentiable at $x = x_1$ does not cause any problem because the state trajectory $x_\alpha(t; z)$ satisfies $x_\alpha(t; z) = x_1$ at most at a single point in time. □

Parts (a) and (b) of Theorem 1 are therefore established. Part (c) follows easily from $\phi_\alpha(x) < F(x)/n$ for all $x \in (0, \bar{x}_\alpha)$ and $\phi_\alpha(x) > F(x)/n$ for all $x \in (\bar{x}_\alpha, 1]$. To see that part (d) holds just note that by choosing α sufficiently close to α_0 we can make $\phi_\alpha(x_1)$ arbitrary close to $\alpha_0 = F(x_1)/n$. Therefore we get $\dot{x}_\alpha(t; z)|_{x_\alpha(t;z)=x_1} = F(x_1) - n\phi_\alpha(x_1) \approx 0$ so that $x_\alpha(t; z)$ will spend an arbitrary long time in the vicinity of x_1. This completes the proof of Theorem 1.

3.5 Proof of Theorem 2: The proof that (5) is sufficient is very similar to the content of subsection 3.3 and we shall omit many details. As in 3.3 one obtains the differential equation (18). This time, however, we choose (10) as the initial condition. This yields

$$V(x)=\frac{U(k)}{r}-\frac{U(k)-U(0)}{r}\exp\left[-r\int_0^x\frac{dy}{nk-F(y)}\right].$$

Because of this construction, (8) is automatically satisfied. Condition (7) can again be verified by using the first order condition $U'(k)\geq V'(x)$ which is equivalent to

$$U'(k)\geq\frac{U(k)-U(0)}{nk-F(x)}\exp\left[-r\int_0^x\frac{dy}{nk-F(y)}\right].$$

Since the right hand side attains its unique maximum at $x=x_1$ it follows that (5) is sufficient for the first order condition to hold and, hence, for the fact that maximal exploitation is a Markov-perfect Nash equilibrium.

It will be shown below (part (b) of Theorem 2) that condition (5) holds automatically if $x_1=0$. To prove the necessity of (5) we may therefore restrict ourselves to the case $x_1>0$. The basic idea is to use a variational argument. Assume that the second condition of Theorem 2(a) is true. This means that the policy function defined in (6) is an optimal feedback solution to the optimal control problem

$$\begin{array}{ll}\text{Maximize} & \displaystyle\int_0^\infty e^{-rt}U(c(t))\,dt\\ \text{subject to} & \dot{x}(t)=F(x(t))-(n-1)k-c(t),\\ & c(t)\in C(x(t))\,,\ x(0)=z\end{array}$$

where z can be any initial state in $[0,1]$. Let us choose the particular initial state $z=x_1$. Now consider the alternative policy function $\tilde{\phi}(x)=\phi(x)-\epsilon h(x)$ where $\epsilon>0$ and $h:[0,1]\mapsto[0,\infty)$ is any non-negative smooth function satisfying $h(0)=0$. It is obvious that $\tilde{\phi}(x)\in C(x)$ holds for all $x\in[0,1]$ provided that ϵ is chosen sufficiently small. Let us fix the function h and denote by $x_\epsilon(t)$ the state trajectory generated by the policy function $\tilde{\phi}$ from the initial state $z=x_1$. In other words, as long as $x_\epsilon(t)$ is positive it satisfies the initial value problem

$$\dot{x}_\epsilon(t)=F(x_\epsilon(t))-nk+\epsilon h(x_\epsilon(t))\,,\ x_\epsilon(0)=x_1.$$

Because of assumption A3 it is clear that $x_\epsilon(t)$ will become zero within finite time $T(\epsilon)$ provided ϵ is small. Furthermore, from the above differential equation it follows that $T(\epsilon)$ is given by

$$T(\epsilon) = \int_0^{x_1} \frac{dy}{nk - F(y) - \epsilon h(y)}. \tag{21}$$

The value of the path generated by the policy function $\tilde{\phi}$ and the initial condition $z = x_1$ is therefore $V(\epsilon) = \int_0^{T(\epsilon)} e^{-rt} U(k - \epsilon h(x_\epsilon(t)))\, dt + e^{-rT(\epsilon)} U(0)/r$. Since, by assumption, $\epsilon = 0$ corresponds to an optimal path, we must have $V'(0) \leq 0$. It is straightforward to verify that

$$V'(0) = T'(0) e^{-rT(0)} [U(k) - U(0)] - \int_0^{T(0)} e^{-rt} U'(k) h(x_0(t))\, dt$$

and

$$T'(0) = \int_0^{x_1} \frac{h(y)}{[nk - F(y)]^2}\, dy = \int_0^{T(0)} \frac{h(x_0(t))}{nk - F(x_0(t))}\, dt.$$

Combining these two equations we obtain

$$V'(0) = \int_0^{T(0)} h(x_0(t)) \left[e^{-rT(0)} \frac{U(k) - U(0)}{nk - F(x_0(t))} - e^{-rt} U'(k) \right] dt.$$

Now recall that h was an arbitrary non-negative smooth function and that $T(0) > 0$ because $x_1 > 0$. The necessary condition $V'(0) \leq 0$ can therefore only hold if the term inside the brackets on the right hand side of the above equation is non-positive for all $t \in [0, T(0)]$. In particular, for $t = 0$ we obtain because of $x_0(0) = x_1$ that $U'(k) \geq e^{-rT(0)} [U(k) - U(0)]/[nk - F(x_1)]$ must hold. Substituting for $T(0)$ from (21) one gets condition (5). This completes the proof of Theorem 2(a).

To prove (b) we first note that $r \geq F'(0)$ is equivalent to $x_1 = 0$. Therefore, (5) is equivalent to $U'(k) \geq [U(k) - U(0)]/(nk)$. The validity of this inequality follows from Lemma B in the appendix by letting α approach 0.

The proof of assertion (c) is straightforward and uses the fact that the right hand side of (5) converges to 0 as n approaches infinity.

Finally, to prove (d) rewrite (5) as $1 \geq f(k)g(k)$ with $f(k) = [U(k) - U(0)]/[nkU'(k)]$ and

$$g(k) = \frac{nk}{nk - F(x_1)} \exp\left[-r \int_0^{x_1} \frac{dy}{nk - F(y)} \right].$$

Using de rule of de l'Hopital one can show that $\lim_{k\nearrow\infty} f(k) = [n(1 - \lim_{c\nearrow\infty}\beta(c))]^{-1}$. Furthermore, $\lim_{k\nearrow\infty} g(k) = 1$. Taking all these properties together it follows immediately that (5) is satisfied for all sufficiently large k. This completes the proof of Theorem 2. □

4 Concluding remarks

In this paper we have considered a very general model of the joint exploitation of a renewable resource by a finite number of non-cooperating agents. We have focussed on the existence, (non-)efficiency, and multiplicity of Markov-perfect Nash equilibria. Our main results are

- that there exist infinitely many equilibria which lead to a tragedy of the commons provided the resource is sufficiently productive,
- that one can approximate with arbitrary high precision the efficient steady state by state trajectories generated by these equilibria provided that the initial stock of the resource is sufficiently high,
- and that the situation where all agents exploit the resource at maximal rate qualifies as an equilibrium provided that there are either very many agents, very impatient agents, or very effective agents.

Although these results seem to be very comprehensive they probably raise more questions than they answer. Which equilibrium will be selected out of the infinite set of equilibria? How can agents coordinate on those equilibria that are more efficient? How can agents prevent getting trapped in the most rapid extinction equilibrium? Are there even more Markov-perfect equilibria in this model, perhaps equilibria resulting in underconsumption (as in [7, 8])? We believe that answering these questions is essential for a better understanding of renewable resource markets. At the present moment, however, we are still unable to say anything substantial about these issues.

Apart from these fundamental questions there are also some other (more technical) open problems which we propose as topics for future research. First of all, our analysis is not complete in the sense that the existence conditions (4) and (5) do not cover all possible cases. This has been demonstrated by the example at the end of Section 2. Other open problems concern the major structural assumptions of the present paper, A1 and A5. Although assumption A1 is frequently imposed in models of common property resource extraction it is not undisputed. As a matter of fact, depending on the type of resource under consideration it may be more realistic to consider non-concave growth functions F. The cases of depensation ($F(x)$ is convex for small x, concave for large x, and strictly positive for all $x \in (0,1)$) and critical depensation ($F(x)$ is convex for small x, concave for large x, negative for small x, and positive for large x) are of particular interest (see [2, p. 17]). In these cases the results of Theorem 1 will most likely fail. Markov-perfect Nash equilibria, if they exist at all, can be conjectured to have a structure which is quite different from the equilibria in Theorem 1. In particular, one would conjecture that already in the case of depensation there exists a positive stock level $\hat{x}$ such that extinction is ineviatable for all initial stocks smaller than $\hat{x}$. This conjecture is motivated by the results for the one-player version of model (1) - (3) with depensation which is analyzed in [12].[13] As for assumption A5 we do not know how its relaxation would effect the results of the present paper. It is used at two different places in the analysis: to ensure that the vector field defined by (12) is pointing upwards and in Lemma B in the appendix. We believe, however, that the structure of equilibria would be affected quite dramatically if one were to relax A5. Finally, we think that the analysis of the game with asymmetric players would be a worthwhile project.

Appendix

First we prove a simple result which is used at two different points in the paper.

Lemma A *Let $f : (0, \bar{x}] \mapsto I\!R$ be a continuously differentiable and strictly positive function and assume that there exist real constants δ and K such that the inequality*

$$-xf'(x)/f(x) \leq \delta + Kx \tag{22}$$

holds for all $x \in (0, \bar{x}]$. Then there exists a real constant L such that $f(x) \leq Lx^{-\delta}$ holds for all $x \in (0, \bar{x}]$. If the inequality in (22) is reversed then it follows that $f(x) \geq Lx^{-\delta}$ for all $x \in (0, \bar{x}]$.

[13] In the case of critical depensation the existence of a stock size below which extinction is inevitable is obvious.

PROOF. Inequality (22) can be written as

$$\frac{d}{dx}\left[\ln f(x)\right] \geq \frac{d}{dx}\left[\ln x^{-\delta} - Kx\right].$$

Integration over the interval $[x, \bar{x}]$ and exponentiation yields $f(\bar{x})/f(x) \geq (x/\bar{x})^{\delta} e^{K(x-\bar{x})}$. Defining the constant L by $L = f(\bar{x})\bar{x}^{\delta} \max\{e^{K\bar{x}}, 1\}$ it is easily seen that this inequality implies $f(x) \leq Lx^{-\delta}$. This proves the first assertion of the lemma. The second one can be shown analogously. □

The following result is used in the proof of Lemma 5.

Lemma B *Let $U : [0, k] \mapsto I\!R$ be a utility function satisfying assumptions A2 and A4. For all $\alpha \in (0, k)$ it holds that*

$$U'(k) > \frac{U(k) - U(\alpha)}{n(k - \alpha)}. \tag{23}$$

PROOF. From A4 it follows that $U''(c)/U'(c) = (d/dc) \ln U'(c) = -\beta(c)/c$ for all $c \in (0, k]$. Integration over the interval $[\alpha, x]$ yields

$$U'(x) = U'(\alpha) \exp\left[-\int_{\alpha}^{x} \frac{\beta(c)}{c}\, dc\right] \tag{24}$$

for all $x \in [\alpha, k]$. Integrating this equation once more over the interval $[\alpha, k]$ yields

$$U(k) - U(\alpha) = U'(\alpha) \int_{\alpha}^{k} \exp\left[-\int_{\alpha}^{x} \frac{\beta(c)}{c}\, dc\right] dx.$$

Using (24) with $x = k$ we see that this equation can also be written as

$$U(k) - U(\alpha) = U'(k) \int_{\alpha}^{k} \exp\left[\int_{x}^{k} \frac{\beta(c)}{c}\, dc\right] dx.$$

By substituting this on the right hand side of (23) we see that (23) is equivalent to

$$n(k - \alpha) > \int_{\alpha}^{k} \exp\left[\int_{x}^{k} \frac{\beta(c)}{c}\, dc\right] dx. \tag{25}$$

Because $\beta(c) \leq (n-1)/n$ holds by assumption A4 a sufficient condition for (25) is given by

$$n(k - \alpha) > \int_{\alpha}^{k} \exp\left[\int_{x}^{k} \frac{n-1}{nc}\, dc\right] dx.$$

The integrals on the right hand side can be computed which shows that the inequality is equivalent to $n(k-\alpha) > nk^{(n-1)/n}\left(k^{1/n} - \alpha^{1/n}\right)$. Simple rearrangments show that this inequality is indeed true for all $\alpha < k$. $\square$

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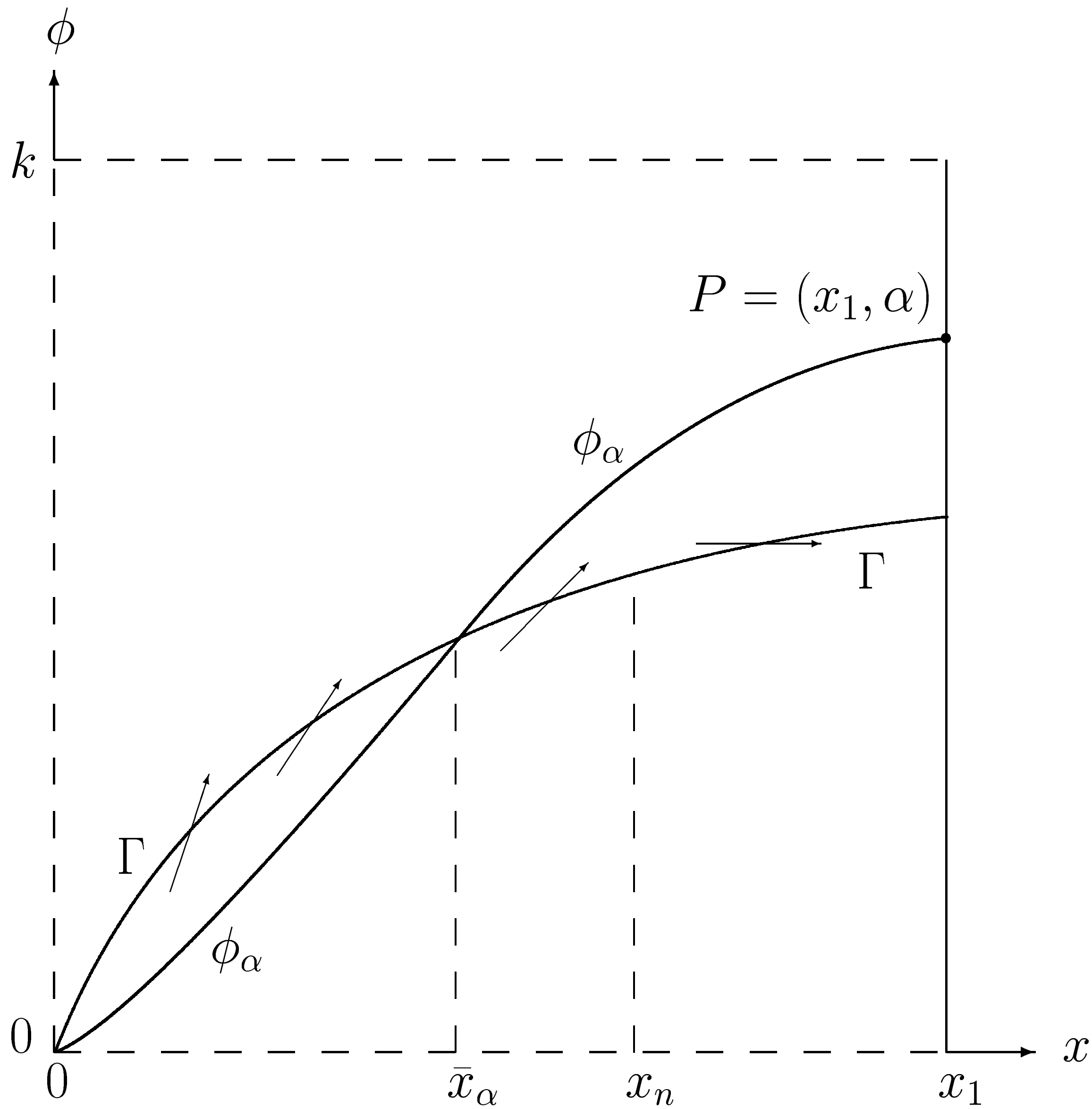

Figure 1: The domain D and the curve ,

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