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NONPARAMETRIC METHODS FOR COMPARING DISTRIBUTION FUNCTIONALS FOR DEPENDENT SAMPLES WITH APPLICATIONS TO WELFARE INDICES

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Nonparametric methods for comparing distribution functionals for dependent samples with applications to welfare indices[*]

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Abstract

This paper proposes asymptotically distribution-free inference methods for comparing estimators which admit asymptotically linear Gaussian functional representations across dependent samples. The framework applies to a broad range of welfare indices used in inequality, poverty, and risk analysis. Two distinct situations are considered. First, we propose asymptotic and bootstrap intersection methods which are valid under arbitrary dependence between two samples. Second, we focus on the common case of overlapping samples—a special form of dependent samples where sample dependence arises solely from matched pairs—and provide asymptotic and bootstrap methods for comparing indices. We derive consistent estimates for asymptotic variances using the influence function approach. We study the finite-sample performance of the proposed methods through Monte Carlo simulations and find that confidence intervals based on overlapping samples exhibit satisfactory coverage rates and reasonable precision. In contrast, conventional methods based on the assumption of independent samples perform poorly in terms of coverage rates and interval widths. Asymptotic inference can be less reliable when dealing with heavy-tailed distributions, while the bootstrap method provides a viable remedy, unless the variance is substantial or fails to exist. The intersection method yields reliable results with arbitrary dependent samples, including settings in which the overlapping-sample assumptions do not hold. We demonstrate the practical applicability of our proposed methods in analyzing changes in household financial inequality in Italy over time.

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Résumé

Cet article propose des méthodes d’inférence asymptotiquement indépendantes de la distribution pour comparer des estimateurs qui admettent des représentations fonctionnelles gaussiennes asymptotiquement linéaires sur des échantillons dépendants. Ce cadre s’applique à un large éventail d’indices de bien-être utilisés dans l’analyse des inégalités, de la pauvreté et des risques. Deux situations distinctes sont examinées. Premièrement, nous proposons des méthodes d’intersection asymptotiques et par bootstrap qui sont valables en présence d’une dépendance arbitraire entre deux échantillons. Ensuite, nous nous concentrons sur le cas courant des échantillons qui se chevauchent — une forme particulière d’échantillons dépendants où la dépendance résulte uniquement de paires appariées — et proposons des méthodes asymptotiques et de bootstrap pour comparer les indices. Nous dérivons des estimations cohérentes des variances asymptotiques à l’aide de l’approche par la fonction d’influence. Nous étudions les performances en échantillon fini des méthodes proposées à l’aide de simulations de Monte Carlo et constatons que les intervalles de confiance basés sur des échantillons chevauchants présentent des taux de couverture satisfaisants et une précision raisonnable. En revanche, les méthodes conventionnelles reposant sur l’hypothèse d’échantillons indépendants affichent de mauvaises performances en termes de taux de couverture et de largeurs d’intervalle. L’inférence asymptotique peut s’avérer moins fiable lorsqu’il s’agit de distributions à queues épaisses, tandis que la méthode du bootstrap offre une solution viable, à moins que la variance ne soit importante ou n’existe pas. La méthode d'intersection fournit des résultats fiables pour des échantillons dépendants arbitraires, y compris dans les cas où les hypothèses relatives au chevauchement des échantillons ne sont pas vérifiées. Nous démontrons l'applicabilité pratique des méthodes que nous proposons en analysant l'évolution des inégalités financières entre les ménages en Italie au fil du temps.

Keywords/Mots-clés: inequality measures; poverty measures; influence function; asymptotic inference; intersection method; bootstrap; confidence interval; overlapping samples; dependent samples / mesures d'inégalité ; mesures de pauvreté ; fonction d'influence ; inférence asymptotique ; méthode d'intersection ; bootstrap ; intervalle de confiance ; échantillons qui se chevauchent ; échantillons dépendants

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1. Introduction

Researchers often seek to compare inequality levels across different areas or over time. In practice, inequality measures are estimated from samples, making it essential to investigate the robustness of these comparisons through statistical inference. However, most inference procedures assume either independence between samples or complete dependence (matched pairs of observations), as detailed in Mills and Zandvakili (1997), Cowell and Flachaire (2015), Dufour, Flachaire and Khalaf (2019), Dufour, Flachaire, Khalaf and Zalghout (2024), Ibragimov, Kattuman and Skrobotov (2025), and references therein.

Nevertheless, many income data sets are dependent because there is an overlap between the two samples. Overlap occurs when the samples contain matched observations, such as the incomes of the same household interviewed in two consecutive periods. For instance, labor force surveys in many countries are, by design, rotated panel data. Other examples include the Current Population Survey (CPS), the Panel Study of Income Dynamics (PSID), and the Consumer Expenditure Survey (CEX) for U.S. income data. In this paper, we utilize the Survey of Household and Income Wealth (SHIW) from Italy, featuring an overlap between consecutive waves. Two issues emerge when using such data: first, some individual income observations are correlated across periods; second, income is unobserved for some units in one of the periods.

Several studies have provided inference procedures with overlapping samples. Zheng and J. Cushing (2001) proposed an asymptotic method for comparing interpolated Gini indices and generalized entropy measures. Zheng (2004) extended these results to the Foster-Greer-Thorbecke poverty measure. Biewen (2002) presented a percentile bootstrap for comparing indices in a family of smooth functions of population moments. The above studies focus on specific indices and do not cover the popular Gini index and quantile-based functionals, such as the Lorenz curve. Moreover, some indices are covered by only one of the two methods (*i.e.*, the asymptotic and bootstrap methods).

We focus on estimators which can be represented as asymptotically linear Gaussian (ALG) functionals (defined in the next section). These include a broad class of inequality, poverty, and risk measures within a unified framework. ALG-type estimators in this context have been considered in Dufour et al. (2019, henceforth DFK). Nevertheless, we differ from DFK in two important ways. First, DFK focus on hypothesis tests based on permutation methods, whereas our approach does not. Second, DFK assume independent samples, while we allow for general forms of dependence between the samples compared. We consider a broader family of indices, including quantile-based measures such as the Lorenz curve.

Our methods also build on the literature on influence functions (IFs), dating back to Hampel (1974). As one of the earliest applications in econometrics, Cowell and Victoria-Feser (1996) investigated the robustness of various inequality measurements. In contrast, we employ the IF to derive the asymptotic distribution and estimate the asymptotic variance with substantial computational advantage; see, for example, Cowell and Flachaire (2015). Furthermore, this paper extends the estimation of asymptotic variances to dependent samples. We make the following contributions.

First, we consider a wide range of indices covering most inequality, poverty, and risk-analysis measures, including the well-known Gini index and Lorenz curve.

Second, we develop methods applicable without restrictions on cross-sample dependence. This is done by using the intersection approach previously considered by Dufour and Torrès (1998) in a finite-sample framework. We introduce asymptotic and bootstrap intersection methods (IMs) for assessing changes in indices. These methods utilize information within each sample and handle arbitrary dependence between two samples. Furthermore, our method provides solutions to a generalized Behrens-Fisher problem, which involves comparing the means of two distributions with arbitrary dependence across the two samples.

Third, we examine overlapping samples, a specific form of dependent samples where sample dependence arises solely from overlap (*i.e.*, matched pairs). Such a framework is applicable to split or rotating panels in survey data. We propose asymptotic and bootstrap inference for index differences, and we present a consistent and numerically nonnegative variance estimator for scalar index differences, and discuss the positive-semidefiniteness issue for vector-valued cases.

Fourth, we conduct a series of simulation experiments to examine the performance of the proposed methods. Our confidence intervals based on overlapping samples have coverage rates close to the nominal level with reasonable widths. By contrast, conventional inference—which ignores sample dependence—can fail to cover the true value with adequate frequency or become overly conservative with large widths. We then analyze the impacts of heavy tails and find that the asymptotic inference performs well for realistic distributions but poorly for distributions with heavy tails. The bootstrap method can alleviate this issue, except when the variance is substantial or fails to exist. We show that intersection methods can yield reliable results across the scenarios considered, particularly when the overlapping-sample assumptions do not hold, and can be reasonably efficient in some instances, such as extremely heavy-tailed distributions.

Fifth, we apply the proposed methods to investigate changes in financial inequality from 2012 to 2014. The results reveal distinct patterns of internal inequality within winner/loser subgroups, highlighting the practical importance of negative values in inequality measurement.

Sixth, we show that the proposed methodology, though motivated by welfare analysis, is considerably more general, and applies to any estimator with an asymptotically linear Gaussian expansion. Welfare applications are emphasized because they are intrinsically important and provide natural examples.

The paper is organized as follows. Section 2 reviews commonly used asymptotically linear welfare indices. Section 3 describes inference based on intersection methods. Section 4 discusses the special case of overlapping samples. We assess the performance of the proposed procedures through Monte Carlo experiments in Section 5. In Section 6, we apply the methods to analyze the change in household financial inequality in Italy. We briefly discuss the extension to clustered data to broaden the scope of our analysis in Section 7. The paper concludes in Section 8. All proofs are given in the appendix.

2. Asymptotically linear Gaussian functionals

To achieve an appropriate level of generality, we consider functional parameters whose estimators admit asymptotically linear Gaussian expansions. This class encompasses estimators of many welfare measures and provides a convenient way of deriving the relevant asymptotic distributional theory.

To facilitate our discussion, we introduce some notation. Let $F \in \mathscr{F}$ be a cumulative distribution function (CDF), and $\theta : \mathscr{F} \to \mathbb{R}$ a functional of distribution F. For a sample $\{X_i\}_{i=1}^n$, we denote by $\hat{F}$ the empirical distribution function (EDF) for F, and estimate $\theta(F)$ by $\theta(\hat{F})$. We denote by $o_p(1)$ a sequence that converges (in probability) to zero as $n \to \infty$ and write $X_n \xrightarrow{d} X$ for convergence in distribution. We consider a random sample of independent and identically distributed (iid) observations $\{X_i\}_{i=1}^n$ drawn from the distribution F, written as $\{X_i\}_{i=1}^n \overset{iid}{\sim} F$. We now define asymptotically linear Gaussian (ALG) functionals.

Definition 2.1 ASYMPTOTICALLY LINEAR GAUSSIAN FUNCTIONAL. *Given a sample $\{X_i\}_{i=1}^n$ drawn from a distribution F, an estimator $\theta(\hat{F})$ for the functional $\theta(F)$ is* asymptotically linear Gaussian *if it satisfies*

$$\sqrt{n}[\theta(\hat{F}) - \theta(F)] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(X_i; \theta, F) + o_p(1) \xrightarrow[n\to\infty]{d} \mathrm{N}[0, \sigma_\theta^2(F)] \tag{2.1}$$

where $\sigma_\theta^2(F) := \mathrm{Var}[\psi(X_i; \theta, F)]$.

In many cases, the function ψ in (2.1) is the (asymptotic) influence function (IF) of the estimator $\theta(\hat{F})$, which is defined as follows:

$$\psi(x; \theta, F) = \lim_{t\to 0} \frac{\theta((1-t)F + t\delta_x) - \theta(F)}{t} \tag{2.2}$$

where δ_x denotes a point mass at x. The IF has been widely used to analyze the robustness of an estimator, as it quantifies the effect on an estimator when a single data point in the sample is contaminated; see Wasserman (2006) for further details. Here, we utilize the IF as a convenient device to derive asymptotic distributions, especially the asymptotic covariance matrices.

Many estimators admit ALG expansions: sample estimates of moments such as skewness and kurtosis, and functions of moments such as the Sharpe ratio (Zhang, Martin and Christidis (2021)).[1] Definition 2.1 relies on a high-level condition, but in practice, we often impose more primitive assumptions.

A leading application of our procedures is inference on welfare indices. Table 1 lists ALG estimators for popular indices and associated references. For illustrative purposes, we present examples of the Gini index and the Lorenz curve (LC) below. The key requirement is not that the object be a welfare index, but that its estimator admit the expansion in Definition 2.1 under the maintained regularity conditions.

Example 2.1 GINI INDEX. The Gini index, also known as the Gini coefficient, measures the degree of inequality using a scalar value ranging from 0 to 1, where 0 represents perfect equality

[1] The framework does not automatically cover all empirical inequality or welfare procedures. For example, non-regular or non-differentiable functionals, estimators converging more slowly than $\sqrt{n}$, and procedures defined through numerical optimization or tuning-parameter-dependent smoothing may require separate analysis. We therefore restrict attention to cases where an influence-function-based asymptotically linear representation is available.

Table 1
List of estimators for welfare indices as asymptotically linear Gaussian functionals

	Welfare indices	References
Inequality	Gini index, Lorenz curve, generalized Lorenz curve, generalized entropy class, Atkinson index, coefficient of variation, Pietra ratio	Cowell and Flachaire (2015)
	S-Gini index, E-Gini index	Barrett and Donald (2009)
	Extended Lorenz curves, extended Gini indices	Dufour and He (2024*b*)
	Quantile ratio index	Scarpa, Ferrante and Sperlich (2025)
Poverty	Foster-Greer-Thorbecke poverty measure, Sen poverty index, Sen-Shorrocks-Thon Poverty Index	Cowell and Flachaire (2015)
	Gini poverty index	Barrett and Donald (2009)
Risk measurement	Value-at-Risk, expected shortfall	Zhang et al. (2021)

Note— The table displays a collection of estimators that satisfy Definition 2.1. Relevant conditions and the expressions of the influence functions are given in the referenced papers.

and 1 indicates complete inequality. For an income variable $X \geq 0$ with distribution F, there are over a dozen formulas for the Gini index; see, for instance, Yitzhaki and Schechtman (2013). We will use the following definition:

$$I(F) = \frac{2\int tF(t)dF(t)}{\mu(F)} - 1. \tag{2.3}$$

One can estimate $I(F)$ by plugging in the EDF $\hat{F}$, which yields

$$I(\hat{F}) = \frac{\frac{2}{n^2}\sum_{i=1}^{n}(i-0.5)X_{(i)}}{\mu(\hat{F})} - 1 \tag{2.4}$$

where $X_{(1)} \leq \cdots \leq X_{(n)}$ are the ordered statistics from the sample $\{X_i\}_{i=1}^{n}$. Davidson (2009) has proposed a bias-corrected estimate $\widetilde{I} = \left(\frac{n}{n-1}\right) I(\hat{F})$. The author then applies the delta method to demonstrate that both estimators satisfy Definition 2.1 with $\psi(x;I,F) = h(x;I,F) - \mathbb{E}[h(X;I,F)]$, where

$$h(x;I,F) = \frac{2xF(x) - 2\int t\mathbf{1}(t<x)\,dF(t) - [I(F)+1]x}{\mu(F)}. \tag{2.5}$$

We can estimate this function by substituting the EDF $\hat{F}$, which can be efficiently computed based on order statistics as follows.

$$h\big(X_{(i)};I,\hat{F}\big)=\frac{\left(\frac{2i}{n}\right)X_{(i)}-\frac{2}{n}\sum_{j=1}^{i}X_{(j)}-(\hat{I}+1)X_{(i)}}{\frac{1}{n}\sum_{i=1}^{n}X_i}. \tag{2.6}$$

Example 2.2 LORENZ CURVE. Consider an income variable $X\geq 0$ with distribution F. The Lorenz curve (LC) is defined as follows [Gastwirth (1971)]:

$$L(p;F)=\frac{\int_0^p Q(u;F)\,du}{\mu(F)} \tag{2.7}$$

where $\mu(F)=\int x\,dF>0$ is the mean and $Q(p;F)=\inf\{x:F(x)\geq p\}$ is the p-th quantile. The equality $L(p)=m$ means that the $100p\%$ poorest households receive $100m\%$ of the total income. We obtain the following estimator by substituting the EDF $\hat{F}$:

$$L\big(p;\hat{F}\big)=\frac{\frac{1}{n}\sum_{i=1}^{n}X_i\mathbf{1}\big[X_i\leq Q\big(p;\hat{F}\big)\big]}{\mu\big(\hat{F}\big)}. \tag{2.8}$$

Beach and Davidson (1983) and Cowell and Flachaire (2015) showed by the delta method that, if the distribution F is differentiable at its p-th quantile, then $L\big(p;\hat{F}\big)$ satisfies Definition 2.1 with $\psi(x;L,F,p)=h(x;L,F,p)-\mathbb{E}[h(X;L,F,p)]$, where

$$h(x;L,F,p)=\frac{\big[x-Q(p;F)\big]\mathbf{1}\big[x\leq Q(p;F)\big]-xL(p;F)}{\mu(F)}. \tag{2.9}$$

The ALG representation (2.1) can be established in several ways. For instance, Davidson (2009) derive the ALG expansion by delta method. However, one can also achieve results by using more advanced techniques, such as empirical process theory [Davidson (2010)]. Likewise, the S-Gini [Donaldson and Weymark (1980)] and E-Gini [Chakravarty (1988)], which are two popular extensions of the Gini index, satisfy Definition 2.1 using either the delta method [Davidson (2010)] or empirical process theory [Barrett and Donald (2009)]. In this paper, we only assume the existence of an ALG form as defined in Definition 2.1, thereby avoiding various complex conditions.

One merit of the ALG representation is that it facilitates estimation of the asymptotic variances of the indices. Specifically, we can estimate the asymptotic variance by its sample analog:

$$\hat{\sigma}^2:=\sigma_\theta^2(\hat{F})=\frac{1}{n}\sum_{i=1}^{n}\psi^2(X_i;\theta,\hat{F}) \tag{2.10}$$

where $\hat{F}$ denotes the EDF and ψ is given in (2.1). Barrett and Donald (2009) has validated the consistency of $\hat{\sigma}^2$; see Section 4.2 for a more general scenario.

In the next two sections, we derive asymptotic inference methods for index differences under dependent samples, when both sample sizes go to infinity. In Section 3, we consider generic dependent samples, where we impose no restrictions on the dependence between the two samples or the joint limiting distribution (if it exists). Our proposed inference method is therefore applicable and robust to a wide range of scenarios. Then, in Section 4, we consider a more specific data structure called overlapping samples, where the dependence arises from matched pairs.

We use the following notation. For distribution F_k, $k = 1,2$, we estimate the index $\theta_k = \theta_k(F_k)$ by $\hat{\theta}_k = \theta(\hat{F}_k)$, where $\hat{F}_k$ is the EDF for F_k based on $\{X_{ki}\}_{i=1}^{n_k}$. We denote by $\Delta\hat{\theta} = \hat{\theta}_1 - \hat{\theta}_2$ the estimator of the index difference $\Delta\theta = \theta_1 - \theta_2$. We wish to make inference on $\Delta\theta$. This problem is a nonparametric version of the Behrens-Fisher problem, which involves comparing means of potentially distinct distributions [Lehmann and Romano (2022, p 846)]. We will also consider the possibility of dependence between the two samples.

3. Intersection methods for generic dependent samples

We start with generic dependent samples, where the observations in one sample may not be independent of those of the other sample. For simplicity, the observations within each sample are assumed to be independent and identically distributed (iid). However, it is straightforward to extend results to clustered data using techniques by Zheng (2002), Bhattacharya (2007), and Ibragimov et al. (2025).

Assumption 3.1 GENERIC DEPENDENT SAMPLES. *Assume that* $\{X_{1i}\}_{i=1}^{n_1} \overset{iid}{\sim} F_1$ *and* $\{X_{2i}\}_{i=1}^{n_2} \overset{iid}{\sim} F_2$.

Assumption 3.1 only focuses on individual samples $\{X_{1i}\}_{i=1}^{n_1}$ and $\{X_{2i}\}_{i=1}^{n_2}$, allowing arbitrary forms of dependence between the two samples. The generality of this assumption will be more apparent when we introduce overlapping samples in the subsequent section.

3.1. Asymptotic intersection method

As mentioned in (2.10), $\hat{\sigma}_k^2$ is a consistent estimator for the asymptotic variance of $\hat{\theta}_k$, where

$$\hat{\sigma}_k^2 := \sigma_\theta^2(\hat{F}_k) = \frac{1}{n_k}\sum_{i=1}^{n_k} \psi^2(X_{ki};\theta,\hat{F}_k), \quad k = 1,2. \tag{3.11}$$

Let q_p be the p-th quantile of the standard Gaussian distribution. We then have the following asymptotic confidence interval (CI) for θ_k at level $(1-\alpha_k)$, denoted by $[L_{n_k}, U_{n_k}]$, where

$$L_{n_k} = \hat{\theta}_k - q_{1-\frac{\alpha_k}{2}}\frac{\hat{\sigma}_k}{\sqrt{n_k}}, \quad U_{n_k} = \hat{\theta}_k + q_{1-\frac{\alpha_k}{2}}\frac{\hat{\sigma}_k}{\sqrt{n_k}}, \quad k = 1,2. \tag{3.12}$$

The limiting confidence coefficient is defined as $\lim_{n_k\to\infty}\inf_{\theta_k\in\Theta_k}\Pr_{\theta_k}(L_{n_k} \le \theta_k \le U_{n_k})$, provided the limit exists (Lehmann and Romano (2022, p 526)).

We propose an asymptotic version of the intersection method (IM) for $\Delta\theta$, adapted from Dufour and He (2024*a*), who extend the results in Dufour and Torrès (1998) to finite-sample inference.

Proposition 3.1 ASYMPTOTIC INTERSECTION METHOD. *Let* $\theta(F_k)$ *be an asymptotically linear Gaussian functional, and* $[L_{n_k}, U_{n_k}]$ *in* (3.12) *an asymptotic CI for* θ_k *with confidence level* $(1-\alpha_k)$ *for* $k = 1,2$. *Then, under Assumption 3.1, the following properties hold*:
(1) $[L_{n_1} - U_{n_2},\ U_{n_1} - L_{n_2}]$ *is an asymptotic CI for* $\Delta\theta$ *with level* $(1-\alpha)$, *where* α_1 *and* α_2 *are such*

that $\alpha \geq \alpha_1 + \alpha_2$;
(2) *if the two samples are independent and* $[L_{n_k}, U_{n_k}]$ *for* $k = 1,2$ *have limiting confidence coefficients* $(1-\alpha_k)$*, then* $[L_{n_1} - U_{n_2},\ U_{n_1} - L_{n_2}]$ *is an asymptotic CI for* $\Delta\theta$ *with confidence level* $(1-\alpha)$*, where* α_1 *and* α_2 *are such that* $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

3.2. Bootstrap intersection method

There are several ways to construct a bootstrap CI for θ_k, $k = 1,2$. We use the Studentized bootstrap, or percentile-t, CI [Hall (1992)]. The procedure is as follows [Davidson and MacKinnon (2004)].

(1) Compute estimates $\hat{\theta}_k$ and $\hat{\sigma}_k$ based on the original samples $\{X_{ki}\}_{i=1}^{n_k}$ for $k = 1,2$.

(2) Resample with replacement from $\{X_{ki}\}_{i=1}^{n_k}$ to construct the bootstrap samples $\{X_{ki}^*\}_{i=1}^{n_k}$. Next, compute the estimates $\hat{\theta}_k^*$ and $\hat{\sigma}_k^*$, and calculate the scaled statistic $T_k^* = \left(\hat{\theta}_k^* - \hat{\theta}_k\right)/\hat{\sigma}_k^*$.

(3) Repeat the previous step B times to obtain the statistics $\{T_{k,j}^*\}_{j=1}^{B}$. Then, compute $q_{k,\frac{\alpha_k}{2}}^*$ and $q_{k,1-\frac{\alpha_k}{2}}^*$, where $q_{k,p}^*$ is the $\lceil pB \rceil$ order statistic of $\{T_{k,j}^*\}_{j=1}^{B}$. Here, $\lceil \cdot \rceil$ denotes the ceiling function.

(4) Build the bootstrap CI for θ_k with confidence level $(1-\alpha_k)$ as $\left[L_{n_k}^B, U_{n_k}^B\right]$, where

$$L_{n_k}^B = \hat{\theta}_k - \hat{\sigma}_k q_{k,1-\alpha_k/2}^*, \quad U_{n_k}^B = \hat{\theta}_k - \hat{\sigma}_k q_{k,\alpha_k/2}^*. \tag{3.13}$$

Ideally, the number of bootstrap samples B should be reasonably large and satisfy the condition that $\alpha(B+1)$ is an integer for any test level α [Davidson and MacKinnon (2000)]. We then propose a bootstrap version of the intersection method for $\Delta\theta$.

Proposition 3.2 BOOTSTRAP INTERSECTION METHOD. *Let* $\theta(F_k)$ *be an asymptotically linear Gaussian functional and* $[L_{n_k}^B, U_{n_k}^B]$ *in* (3.13) *be a bootstrap CI for* θ_k *with confidence level* $(1-\alpha_k)$ *for* $k = 1,2$*. Then, under Assumption 3.1, the following properties hold:*
(1) $\left[L_{n_1}^B - U_{n_2}^B,\ U_{n_1}^B - L_{n_2}^B\right]$ *is a bootstrap CI for* $\Delta\theta$ *with level* $(1-\alpha)$*, where* α_1 *and* α_2 *are such that* $\alpha \geq \alpha_1 + \alpha_2$;
(2) *if the two samples are independent and* $[L_{n_k}^B, U_{n_k}^B]$ *for* $k = 1,2$ *have limiting confidence coefficients* $(1-\alpha_k)$*, then* $\left[L_{n_1}^B - U_{n_2}^B,\ U_{n_1}^B - L_{n_2}^B\right]$ *is a bootstrap CI for* $\Delta\theta$ *with confidence level* $(1-\alpha)$*, where* α_1 *and* α_2 *are such that* $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

Intersection methods (IMs) are asymptotically conservative, which reflects the trade-off between precision and robustness. Recall that the IMs do not require any information on sample dependence or the limiting joint distribution of estimators. The next section demonstrates how to achieve more efficient inference methods under additional assumptions.

4. Overlapping samples

In this section, we focus on a special type of sample known as overlapping samples (OS), which accommodate cross-sample dependence that stems exclusively from the overlap (*i.e.*, matched pairs). Such a structure can arise in rotating or splitting-panels surveys. Several authors,including Zheng and J. Cushing (2001), Zheng (2004), and Biewen (2002), have investigated overlapping samples in their analyses of inequality and poverty and proposed inference methods for specific indices. We describe the OS as follows.

Assumption 4.1 OVERLAPPING SAMPLES. *Let $\{X_{1i}\}_{i=1}^{n_1}$ and $\{X_{2i}\}_{i=1}^{n_2}$ be two samples which satisfy Assumption 3.1. Further, the following conditions are satisfied.*

(A1) The first m observations are matched pairs $\{(X_{1i}, X_{2i})\}_{i=1}^{m}$, where $0 \leq m \leq \min(n_1, n_2)$.

(A2) The matched pairs are drawn from a fixed joint distribution $\widetilde{F}$, whose marginal distributions are F_1 and F_2, i.e. $\{(X_{1i}, X_{2i})\}_{i=1}^{m} \overset{iid}{\sim} \widetilde{F}$.

(A3) The observations that are unmatched in each sample are independent of the observations in the other sample.

- *(a) $\{X_{1i}\}_{i=m+1}^{n_1}$ is independent of $\{X_{2i}\}_{i=1}^{n_2}$.*
- *(b) $\{X_{2i}\}_{i=m+1}^{n_2}$ is independent of $\{X_{1i}\}_{i=1}^{n_1}$.*

(A4) As $n_1, n_2 \to \infty$, we require that $\frac{m}{n_1} \to \lambda_1$, $\frac{m}{n_2} \to \lambda_2$, $\frac{n_1}{n_1+n_2} \to \eta_1$ and $N := \frac{n_1 n_2}{n_1+n_2} \to \infty$, where $\lambda_1, \lambda_2 \in [0,1]$, and $\eta_1 \in [0,1]$.

Condition (A1) assumes that the first m observations in each sample are matched pairs. Furthermore, it implicitly requires that we know which two observations in each sample form a pair. For instance, when comparing income difference between parents and children, the incomes of a parent and a child from the same family make up a pair. If the pairing information is lost, so that it is unclear whether or not the child and parent come from the same household, then such samples will violate (A1).

Condition (A2) assumes that pairs of data are iid observations from a fixed unknown joint distribution $\widetilde{F}$. We may relax such a condition to independent but not identically distributed.

Condition (A3) imposes independence between unmatched data and entire observations in the other sample, indicating that the overlap is the only source of dependence between the two samples.

Condition (A4) clarifies what we mean by asymptotic in OS. First, while both n_1 and n_2 go to infinity, the size of overlap m can be zero or a fixed number, giving $\lambda_1 = \lambda_2 = 0$. Second, n_1 and n_2 can increase to infinity at different rates such that $\eta_1 = 0$ or $\eta_1 = 1$ (and $\lambda_1 = \lambda_2 = 0$). So, we do not view (A4) as a limitation; also see the discussion following Proposition 4.1.

To complete Assumption 4.1, we make an asymptotic normality assumption on matched pairs. The condition for (4.14) is technically necessary because marginal asymptotic normality does not imply joint asymptotic normality.

Assumption 4.2 ASYMPTOTICALLY JOINT NORMALITY. *Let $\hat{\theta}_k$ be an asymptotically linear Gaussian functional with $\psi_k(x) := \psi(x;\theta,F_k)$ in* (2.1) *for $k=1,2$. If $m\to\infty$, then*

$$\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\begin{bmatrix}\psi_1(X_{1i})\\ \psi_2(X_{2i})\end{bmatrix}\xrightarrow[m\to\infty]{d}\mathrm{N}\left[\begin{pmatrix}0\\0\end{pmatrix},\left(\begin{bmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{bmatrix}\right)\right] \tag{4.14}$$

where $\sigma_k^2=\mathrm{Var}[\psi_k(X_{ki})]$ and $\sigma_{12}=\mathrm{Cov}[\psi_1(X_{1i}),\psi_2(X_{2i})]$. Otherwise, we have $\lambda_1=\lambda_2=0$, where λ_1 and λ_2 are given in Assumption 4.1.

Assumption 4.2 is a high-level condition. ALG representations are available in the cited literature for the indices in Table 1. In general, Assumption 4.2 is satisfied when three ingredients are available: (i) each estimator admits the influence-function expansion in Definition 2.1; (ii) the matched observations generate a triangular array for which the vector of matched-pair influence functions, namely $\{(\psi_1(X_{1i}),\psi_2(X_{2i}))\}_{i=1}^m$, obeys a standard multivariate central limit theorem; (iii) the relevant second moments are finite. These conditions cover the main smooth welfare functionals studied here, including the mean, the Gini coefficient, and Lorenz curve. By contrast, functionals lacking a regular influence-function representation, or evaluated at points where differentiability fails, require separate treatment.

4.1. Asymptotic distribution of index difference

We first provide the asymptotic distribution of the estimator for the index difference $\Delta\hat{\theta}$ with overlapping samples.

Proposition 4.1 ASYMPTOTIC METHOD WITH OVERLAPPING SAMPLES. *If $\hat{\theta}_1$ and $\hat{\theta}_2$ are asymptotically linear Gaussian functionals with $\psi_k(x) := \psi(x;\theta,F_k)$ in* (2.1) *for $k=1,2$, then, under Assumptions 4.1 and 4.2, we have*:

$$\sqrt{N}\big(\Delta\hat{\theta}-\Delta\theta\big)\xrightarrow[n_1,n_2\to\infty]{d}\mathrm{N}[0,\sigma_\Delta^2] \tag{4.15}$$

where $\sigma_\Delta^2=(1-\eta_1)\sigma_1^2+\eta_1\sigma_2^2-2\sqrt{\eta_1(1-\eta_1)\lambda_1\lambda_2}\sigma_{12}$, $\sigma_k^2=\mathrm{Var}[\psi_k(X_{k1})]$, $\sigma_{12}=\mathrm{Cov}[\psi_1(X_{11}),\psi_2(X_{21})]$ and $N\to\infty$, η_1, λ_1, $\lambda_2\in[0,1]$ are defined in Assumption 4.1.

The last term of the asymptotic variance in (4.15) indicates that the impact of sample dependence relies on both the overlap portions (*i.e.*, $\lambda_1\lambda_2$) and the covariance between ψ_1 and ψ_2. While we can estimate $\sigma_{12}\equiv\mathrm{Cov}[\psi_1(X_{11}),\psi_2(X_{21})]$ for a given sample, it is challenging to predict the value or even the sign of σ_{12} based solely on $\mathrm{Cov}(X_1,X_2)$. The difficulty arises because the influence function is often a non-monotonic transformation of the data, complicating the preservation of the sign of $\mathrm{Cov}(X_1,X_2)$, as illustrated in Examples 2.1 and 2.2. We will revisit this issue in the simulation experiments.

Assumption 4.1 allows extremely unbalanced samples (where $\eta_1=0$ or 1). Take $\eta_1=1$ as an example. We then have $\lambda_1=\lambda_2=0$, $N\approx n_2\to\infty$, and $\sigma_\Delta^2=\sigma_2^2$, indicating $\hat{\theta}_2$ as the only source of uncertainty. One way to interpret these results is that the sample size n_1 is so much larger than

n_2 that the uncertainty of $\hat{\theta}_2$ dominates that of $\hat{\theta}_1$. In other words, we can treat $\{X_{1i}\}_{i=1}^{n_1}$ as the population and $\hat{\theta}_1$ as the true parameter θ_1 (with negligible sampling error).

We now apply the above proposition to obtain an asymptotic distribution for estimates of differences in Gini indices and LC ordinates.

Example 4.1 GINI INDEX (CONTINUED). Under the conditions of Proposition 4.1, we have:

$$\sqrt{N}\big(\Delta\hat{I} - \Delta I\big) \xrightarrow[n_1,n_2\to\infty]{d} \mathrm{N}\big(0, \sigma_\Delta^2\big) \tag{4.16}$$

where $\sigma_\Delta^2 = (1-\eta_1)\sigma_1^2 + \eta_1\sigma_2^2 - 2\sqrt{\eta_1(1-\eta_1)\lambda_1\lambda_2}\sigma_{12}$, $\sigma_k^2 = \mathrm{Var}[h(X_{k1};I,F_k)]$ for $k=1,2$, and $\sigma_{12} = \mathrm{Cov}[h(X_{11};I,F_1), h(X_{21};I,F_2)]$ with $h(x;I,F)$ given in (2.5).

Example 4.2 LORENZ CURVE (CONTINUED). By Proposition 4.1, we have:

$$\sqrt{N}\big(\Delta\hat{L}(p) - \Delta L(p)\big) \xrightarrow[n_1,n_2\to\infty]{d} \mathrm{N}\big(0, \sigma_\Delta^2(p)\big), \tag{4.17}$$

$$\sigma_\Delta^2(p) = (1-\eta_1)\sigma_1^2(p) + \eta_1\sigma_2^2(p) - 2\sqrt{\eta_1(1-\eta_1)\lambda_1\lambda_2}\sigma_{12}(p), \tag{4.18}$$

where $\sigma_k^2(p) = \mathrm{Var}[h(X_{k1};L,F_k,p)]$ for $k=1,2$, and $\sigma_{12}(p) = \mathrm{Cov}[h(X_{11};L,F_1,p), h(X_{21};L,F_2,p)]$ with $h(x;L,F,p)$ given in (2.9).

We can extend the results to a vector of Lorenz curve ordinates at $\mathbf{p} = (p_1,\ldots,p_s)'$. Let $h(X_{ki};L,F_k,\mathbf{p}) = [h(X_{ki};L,F_k,p_1),\ldots,h(X_{ki};L,F_k,p_s)]'$. The asymptotic variance is

$$\Sigma_\Delta(\mathbf{p}) = (1-\eta_1)\Sigma_1(\mathbf{p}) + \eta_1\Sigma_2(\mathbf{p}) - \sqrt{\eta_1(1-\eta_1)\lambda_1\lambda_2}\big(\Sigma_{12}(\mathbf{p}) + \Sigma_{12}(\mathbf{p})'\big) \tag{4.19}$$

where $\Sigma_k(\mathbf{p}) = \mathrm{Var}[h(X_{ki};L,F_k,\mathbf{p})]$ for $k=1,2$, and $\Sigma_{12}(\mathbf{p}) = \mathrm{Cov}[h(X_{1i};L,F_1,\mathbf{p}), h(X_{2i};L,F_2,\mathbf{p})]$.

4.2. Asymptotic variance estimation by influence functions

To conduct inference on changes in indices, we need a consistent estimator for the asymptotic variance. Let $\hat{\psi}_{n_k}(x) := \psi(x;\theta,\hat{F}_{n_k})$. A natural estimator is

$$\hat{\sigma}_{\Delta,\psi}^2 = \frac{n_2}{n_1+n_2}\hat{\sigma}_{1,\psi}^2 + \frac{n_1}{n_1+n_2}\hat{\sigma}_{2,\psi}^2 - \frac{2m}{n_1+n_2}\hat{\sigma}_{12,\psi}, \tag{4.20}$$

where, for $k=1,2$,

$$\hat{\sigma}_{k,\psi}^2 = \frac{1}{n_k}\sum_{i=1}^{n_k}\hat{\psi}_{n_k}^2(X_{ki}), \qquad \hat{\sigma}_{12,\psi} = \frac{1}{m}\sum_{i=1}^{m}\hat{\psi}_{n_1}(X_{1i})\hat{\psi}_{n_2}(X_{2i}), \tag{4.21}$$

As illustrated in examples of Section 2, the influence function is often written as a demeaned variable. We therefore assume that the influence function has the following form

$$\psi(x;\theta,F) = h(x;\theta,F) - \mathbb{E}[h(X;\theta,F)]. \tag{4.22}$$

To simplify discussion, we denote $h_k(x) := h(x;\theta, F_k)$ and $\hat{h}_{k,n_k}(x) := h(x;\theta,\hat{F}_{n_k})$ for distributions F_k with $k = 1,2$ when no confusion arises. The subscript n_k indicates that the estimated functional is based on sample $\{X_{ki}\}_{i=1}^{n_k}$. Denote the sample mean and variance estimators as follows.

$$\overline{h}_{k,n_k} = \frac{1}{n_k}\sum_{i=1}^{n_k} \hat{h}_{k,n_k}(X_{ki}), \qquad \hat{\sigma}^2_{k,n_k} = \frac{1}{n_k}\sum_{i=1}^{n_k}\left(\hat{h}_{k,n_k}(X_{ki}) - \overline{h}_{k,n_k}\right)^2, \quad k = 1,2. \tag{4.23}$$

A natural sample analog of (4.15) is the following estimator.

$$\widetilde{\sigma}^2_{\Delta} = \frac{n_2}{n_1+n_2}\hat{\sigma}^2_{1,n_1} + \frac{n_1}{n_1+n_2}\hat{\sigma}^2_{2,n_2} - \frac{2m}{n_1+n_2}\hat{\sigma}_{12}, \tag{4.24}$$

where $\hat{\sigma}^2_{k,n_k}$ is the sample variance in (4.23) for $k = 1,2$ and $\hat{\sigma}_{12,m}$ represents the sample covariance of the matched pairs, namely $\hat{\sigma}_{12} = \frac{1}{m}\sum_{i=1}^{m}\left(\hat{h}_{1,n_1}(X_{1i}) - \overline{h}_{1,m}\right)\left(\hat{h}_{2,n_2}(X_{2i}) - \overline{h}_{2,m}\right)$. Indeed, several studies, including Zheng and J. Cushing (2001) and Zheng (2004), have proposed this type of estimator. However, the estimator $\widetilde{\sigma}^2_{\Delta}$ in (4.24) is not guaranteed to be positive (or positive semidefinite for a matrix). Consider an example where $n_1 = n_2 = 2m$, then $\widetilde{\sigma}^2_{\Delta} = \frac{1}{2}(\hat{\sigma}^2_{1,2m} + \hat{\sigma}^2_{2,2m} - 2\hat{\sigma}_{12,m})$. While the sample variances for marginal distributions use the entire data, the estimate $\hat{\sigma}_{12,m}$ only uses m observations, potentially leading to a negative $\widetilde{\sigma}^2_{\Delta}$.

On rewriting the covariance term in (4.15), we get:

$$\sigma^2_{\Delta} = (1-\eta_1)\sigma_1^2 + \eta_1\sigma_2^2 - 2\sqrt{\eta_1(1-\eta_1)\lambda_1\lambda_2}\rho_\theta\sigma_1\sigma_2 \tag{4.25}$$

where $\rho_\theta = \mathrm{corr}(\psi_1(X_{11}), \psi_2(X_{21}))$. We consider the following estimate for the asymptotic variance.

$$\hat{\sigma}^2_{\Delta} = \frac{n_2}{n_1+n_2}\hat{\sigma}^2_{1,n_1} + \frac{n_1}{n_1+n_2}\hat{\sigma}^2_{2,n_2} - \frac{2m}{n_1+n_2}\hat{\rho}_{\theta,m}\hat{\sigma}_{1,n_1}\hat{\sigma}_{2,n_2} \tag{4.26}$$

where $\hat{\sigma}^2_{k,n_k}$ for $k = 1,2$ is given in (4.23), and $\hat{\rho}_{\theta,m}$ is the sample correlation of the matched pairs $\left\{(\hat{h}_{1,n_1}(X_{1i}), \hat{h}_{2,n_2}(X_{2i}))\right\}_{i=1}^{m}$.

To see why $\hat{\sigma}^2_{\Delta}$ in (4.26) is nonnegative, we note:

$$\hat{\sigma}^2_{\Delta} = \frac{n_1 n_2}{n_1+n_2}\left[\frac{\hat{\sigma}^2_{1,n_1}}{n_1} + \frac{\hat{\sigma}^2_{2,n_2}}{n_2} - 2\sqrt{\left(\frac{m}{n_1}\right)\left(\frac{m}{n_2}\right)}\hat{\rho}_{\theta,m}\frac{\hat{\sigma}_{1,n_1}}{\sqrt{n_1}}\frac{\hat{\sigma}_{2,n_2}}{\sqrt{n_2}}\right] \tag{4.27}$$

$$\geq \frac{n_1 n_2}{n_1+n_2}\left[\frac{\hat{\sigma}^2_{1,n_1}}{n_1} + \frac{\hat{\sigma}^2_{2,n_2}}{n_2} - 2\frac{\hat{\sigma}_{1,n_1}}{\sqrt{n_1}}\frac{\hat{\sigma}_{2,n_2}}{\sqrt{n_2}}\right] \tag{4.28}$$

$$= \frac{n_1 n_2}{n_1+n_2}\left(\frac{\hat{\sigma}_{1,n_1}}{\sqrt{n_1}} - \frac{\hat{\sigma}_{2,n_2}}{\sqrt{n_2}}\right)^2 \geq 0. \tag{4.29}$$

An alternative nonnegative estimator of the asymptotic variance is to replace $\hat{\sigma}_{12}$ in (4.24) by

$$\widetilde{\sigma}_{12} = \frac{1}{m}\sum_{i=1}^{m}\left(\hat{h}_{1,n_1}(X_{1i}) - \overline{h}_{1,n_1}\right)\left(\hat{h}_{2,n_2}(X_{2i}) - \overline{h}_{2,n_2}\right). \tag{4.30}$$

Note that $\widetilde{\sigma}_{12}$ uses the means computed from the full marginal samples $\overline{h}_{k,n_k}$ instead of the sample mean of matched observations $(\overline{h}_{1,m}, \overline{h}_{2,m})$ as $\hat{\sigma}_{12}$ does in (4.24). So, the estimator becomes

$$\hat{\omega}_{\Delta}^2 = \frac{n_2}{n_1+n_2}\hat{\sigma}_{1,n_1}^2 + \frac{n_1}{n_1+n_2}\hat{\sigma}_{2,n_2}^2 - \frac{2m}{n_1+n_2}\widetilde{\sigma}_{12} \tag{4.31}$$

where $\hat{\sigma}_{k,n_k}^2$ and $\widetilde{\sigma}_{12}$ are given by (4.23) and (4.30), respectively. To see why $\hat{\omega}_{\Delta}^2$ is nonnegative, denote by $v_{ki} := \hat{h}_{k,n_k}(X_{ki}) - \overline{h}_{k,n_k}$ for $k = 1,2$ and recall $N = \frac{n_1 n_2}{n_1+n_2}$, then we have

$$\begin{aligned}
\hat{\omega}_{\Delta}^2 &= \frac{n_2}{n_1+n_2}\left(\frac{1}{n_1}\sum_{i=1}^{n_1} v_{1i}^2\right) + \frac{n_1}{n_1+n_2}\left(\frac{1}{n_2}\sum_{i=1}^{n_2} v_{2i}^2\right) - \frac{2m}{n_1+n_2}\left(\frac{1}{m}\sum_{i=1}^{m} v_{1i}v_{2i}\right) && (4.32)\\
&= \frac{N}{n_1^2}\left(\sum_{i=1}^{m} v_{1i}^2 + \sum_{i=m+1}^{n_1} v_{1i}^2\right) + \frac{N}{n_2^2}\left(\sum_{i=1}^{m} v_{2i}^2 + \sum_{i=m+1}^{n_2} v_{2i}^2\right) - \frac{2N}{n_1 n_2}\sum_{i=1}^{m} v_{1i}v_{2i} && (4.33)\\
&= N\left[\sum_{i=1}^{m}\left(\frac{v_{1i}}{n_1} - \frac{v_{2i}}{n_2}\right)^2 + \sum_{i=m+1}^{n_1}\left(\frac{v_{1i}}{n_1}\right)^2 + \sum_{i=m+1}^{n_2}\left(\frac{v_{2i}}{n_2}\right)^2\right] \geq 0. && (4.34)
\end{aligned}$$

We provide justification for the consistency of $\hat{\omega}_{\Delta}^2$ in (4.31) and $\hat{\sigma}_{\Delta}^2$ in (4.26) in the Appendix.

Example 4.3 GINI INDEX (CONTINUED). In Example 2.1, we estimate the IF component h and asymptotic variance using order statistics. Since sorting destroys pairing labels, we instead consider estimates based on rank statistics. Given order statistics $X_{k(1)} \leq \cdots \leq X_{k(n_k)}$, we define the rank of X_{ki}, denoted by R_{ki}, as $X_{ki} = X_{k(R_{ki})}$. We estimate h in (2.5) as follows:

$$\hat{h}_{k,n_k}(X_{ki}) := h\big(X_{ki}; I, \hat{F}_{n_k}\big) = \frac{\left(\frac{2R_{ki}}{n_k}\right)X_{ki} - \frac{2}{n_k}\sum_{j=1}^{R_{ki}} X_{k(j)} - \big(\hat{I}_k + 1\big)X_{ki}}{\frac{1}{n_k}\sum_{i=1}^{n_k} X_{ki}}. \tag{4.35}$$

We then estimate the asymptotic variance by (4.26) as follows.

$$\hat{\sigma}_{\Delta}^2 = \frac{n_2}{n_1+n_2}\hat{\sigma}_1^2 + \frac{n_1}{n_1+n_2}\hat{\sigma}_2^2 - \frac{2m}{n_1+n_2}\hat{\rho}_I\hat{\sigma}_1\hat{\sigma}_2 \tag{4.36}$$

where $\hat{\sigma}_k^2$ is the sample variance of $\big\{\hat{h}_{k,n_k}(X_{ki})\big\}_{i=1}^{n_k}$ for $k = 1,2$ and $\hat{\rho}_I$ is the sample correlation coefficient of $\{(\hat{h}_{1,n_1}(X_{1i}), \hat{h}_{2,n_2}(X_{2i}))\}_{i=1}^{m}$.

We can extend $\hat{\omega}_{\Delta}^2$ in (4.31) to estimation for the asymptotic variance matrix for a vector of LC ordinates. One can modify the argument from (4.32) to (4.34) to show the estimator is positive semidefinite (PSD). By contrast, the vector analog of $\hat{\sigma}_{\Delta}^2$ in (4.26) is not automatically PSD without further conditions.

Example 4.4 LORENZ CURVE (CONTINUED). We can extend the Lorenz curve in Example 4.2 to a vector of ordinates at $\mathbf{p}$. Let $\hat{\mathbf{h}}_{k,n_k}(X_{ki}) = h(X_{ki}; L, \hat{F}_k, \mathbf{p})$ for $k = 1,2$ with the sample mean $\overline{\mathbf{h}}_{k,n_k} = \frac{1}{n_k}\sum_{i=1}^{n_k} \hat{\mathbf{h}}_{k,n_k}(X_{ki})$. We estimate $\Sigma_\Delta(\mathbf{p})$ in (4.19) as follows.

$$\hat{\Omega}_{\Delta} = \frac{n_2}{n_1+n_2}\hat{\Omega}_1 + \frac{n_1}{n_1+n_2}\hat{\Omega}_2 - \frac{m}{n_1+n_2}\left(\widetilde{\Omega}_{12} + \widetilde{\Omega}'_{12}\right), \tag{4.37}$$

where

$$\hat{\Omega}_k = \frac{1}{n_k}\sum_{i=1}^{n_k}\left(\hat{\mathbf{h}}_{k,n_k}(X_{ki}) - \overline{\mathbf{h}}_{k,n_k}\right)\left(\hat{\mathbf{h}}_{k,n_k}(X_{ki}) - \overline{\mathbf{h}}_{k,n_k}\right)', \tag{4.38}$$

$$\widetilde{\Omega}_{12} = \frac{1}{m}\sum_{i=1}^{m}\left(\hat{\mathbf{h}}_{1,n_1}(X_{1i}) - \overline{\mathbf{h}}_{1,n_1}\right)\left(\hat{\mathbf{h}}_{2,n_2}(X_{2i}) - \overline{\mathbf{h}}_{2,n_2}\right)'. \tag{4.39}$$

For testing $H_0 : \Delta\theta = c$, we can construct a Student's T-test which, under the null hypothesis, converges in distribution to a standard normal distribution:

$$T = \frac{\sqrt{N}\left(\Delta\hat{\theta} - c\right)}{\hat{\sigma}_{\Delta}} \xrightarrow[n_1,n_2\to\infty]{d} \mathrm{N}(0,1). \tag{4.40}$$

Inverting the test statistic yields an asymptotic confidence interval (CI) of level $(1-\alpha)$ for $\Delta\theta$:

$$\left[\Delta\hat{\theta} - q_{1-\alpha/2}\frac{\hat{\sigma}_{\Delta}}{\sqrt{N}}, \quad \Delta\hat{\theta} + q_{1-\alpha/2}\frac{\hat{\sigma}_{\Delta}}{\sqrt{N}}\right] \tag{4.41}$$

where q_p is the p-th quantile of the standard normal distribution.

4.3. Bootstrap inference with overlapping samples

We now examine the bootstrap method for $\Delta\theta$. Unlike the percentile methods in Mills and Zandvakili (1997) and Biewen (2002), we will explore the Studentized bootstrap, which can benefit from Beran's refinement if the statistic bootstrapped is asymptotically pivotal; see Beran (1988) and Davidson and MacKinnon (2004) for instance. We consider the following procedure to mimic the partial dependence under Assumption 4.1.

1. Given the original samples $\{X_{1i}\}_{i=1}^{n_1}$ and $\{X_{2i}\}_{i=1}^{n_2}$, obtain the estimates $\Delta\hat{\theta}$ and $\hat{\sigma}^2_{\Delta}$ in (4.26), and compute the test statistic T in (4.40).

2. Build a bootstrap sample $\{X^*_{1i}\}_{i=1}^{n_1}$ and $\{X^*_{2i}\}_{i=1}^{n_2}$ as follows. First, draw with replacement from matched pairs $\{(X_{1i}, X_{2i})\}_{i=1}^{m}$ to obtain $\{(X^*_{1i}, X^*_{2i})\}_{i=1}^{m}$. Second, supplement $\{(X^*_{1i}, X^*_{2i})\}_{i=1}^{m}$ by resampling with replacement from the unmatched observations $\{X_{1i}\}_{i=m+1}^{n_1}$ and $\{X_{2i}\}_{i=m+1}^{n_2}$ to obtain $\{X^*_{1i}\}_{i=m+1}^{n_1}$ and $\{X^*_{2i}\}_{i=m+1}^{n_2}$, respectively.

3. For the bootstrap sample $\{X^*_{1i}\}_{i=1}^{n_1}$ and $\{X^*_{2i}\}_{i=1}^{n_2}$, compute the test statistic T^* according to (4.40) with c replaced by $\Delta\hat{\theta}$.

4. Repeat the above step B times and get test statistics $\{T^*_j\}_{j=1}^{B}$. Choose B such that $\alpha(B+1)$ is a positive integer, where α is the nominal level.

5. Compute the bootstrap p-value defined as the proportion of T^*_j that is more extreme than T.

6. Reject the null at level α if the bootstrap p-value is less than α.

We can also construct a two-sided bootstrap CI for $\Delta\theta$ at the level of $(1-\alpha)$ using the empirical distribution of the bootstrap statistics T_j^*:

$$\left[\Delta\hat{\theta} - q^*_{1-\alpha/2}\frac{\hat{\sigma}_\Delta}{\sqrt{N}}, \quad \Delta\hat{\theta} - q^*_{\alpha/2}\frac{\hat{\sigma}_\Delta}{\sqrt{N}}\right] \tag{4.42}$$

where q_p^* is the $\lceil pB \rceil$-th order statistic of $\{T_j^*\}_{j=1}^B$. Here, $\lceil \cdot \rceil$ denotes the ceiling function. The following proposition establishes the validity of the bootstrap method.

Proposition 4.2 BOOTSTRAP METHOD WITH OVERLAPPING SAMPLES. *Let $\Delta\hat{\theta}$ and $\Delta\hat{\theta}^*$ be the estimates based on original and bootstrap samples, respectively. Under conditions of Proposition 4.1, we have*

$$\sqrt{N}\left(\Delta\hat{\theta}^* - \Delta\hat{\theta}\right) \xrightarrow{d} \mathrm{N}\left(0, \sigma_\Delta^2\right) \tag{4.43}$$

where σ_Δ^2 is given in (4.15).

We have two comments. First, while we focus on the bootstrap method using the linearization technique, an alternative approach is the bootstrap method based on U-statistic theory (Shi (1986)). Second, we have treated the overlap count m as fixed in Section 4.3, as we do for the sample sizes n_1 and n_2. Fixing m can be a reasonable assumption for rotating-panel or matched survey designs in which the realized overlap is known at the inference stage.

An alternative view is to treat m, as well as n_1 and n_2, as random by adopting a framework of missing observations. Formally, for $k = 1, 2$, let δ_{ki} be a dummy variable that equals one if we observe X_{ki}, and zero otherwise. We then combine samples into a larger sample of matched pairs as $\{(X_{1i}, \delta_{1i}, X_{2i}, \delta_{2i})'\}_{i=1}^{\bar{n}}$, where $\bar{n} = n_1 + n_2 - m$. Then we employ the pair bootstrap based on $\{\mathbf{Z}_i\}_{i=1}^{\bar{n}}$, where $\bar{n} = n_1 + n_2 - m$. Simulations (not reported here) suggest that the two procedures yield nearly identical results. So, we will not report this type of CI. One may further investigate the formation of matched pairs under additional assumptions, but this is beyond the scope of this paper.

5. Monte Carlo simulations

In this section, we report two simulation experiments on the impacts of sample dependence and heavy tails on various confidence intervals for $\Delta\theta$. For simplicity, we assume that two samples have equal sizes, *i.e.*, $n_1 = n_2 = n$, unless stated otherwise. [2]

Then, we have $\eta_1 = 0.5$, $\lambda_1 = \lambda_2 = \lambda$ in Assumption 4.1 (A4), and the asymptotic standard deviation in Proposition 4.1 reduces to the following quantity:

$$\sigma_\Delta^D = \left(0.5\sigma_1^2 + 0.5\sigma_2^2 - \lambda\rho_\theta\sigma_1\sigma_2\right)^{1/2} \tag{5.1}$$

[2] We also ran experiments for distinct sample sizes, but the results are qualitatively similar to those reported here, though closer to the independent-sample scenario. From the value of σ_Δ^2 in (4.15), this is as expected because the effective contribution of overlap declines. As these additional results do not materially change our conclusions, and to keep the paper concise, we did not add them to the article.

where λ is the overlap portion, $\sigma_k = [\mathrm{Var}(\psi_k(X_{ki}))]^{1/2}$ for $k = 1,2$, and $\rho_\theta = \mathrm{corr}[\psi_1(X_{11}), \psi_2(X_{21})]$ in (4.25).

5.1. Effects of sample dependence

We simulate data from the Singh-Maddala (SM) distribution, which has proven effective in fitting the actual income distributions [McDonald (1984), Kleiber and Kotz (2003)]. The SM distribution [denoted $\mathrm{SM}(b,a,q)$] has distribution function:

$$F(x) = 1 - \frac{1}{[1+(x/b)^a]^q}, \quad a \geq 0, b > 0 \text{ and } q > 1/a, \tag{5.2}$$

where b is a scale parameter, and a and q are shape parameters. The choice of scale parameter b does not matter, as the Gini index and LC are scale-invariant. We assume that the first sample contains data drawn from $\mathrm{SM}(1, 1.6971, 8.3679)$, where the estimates are given by McDonald (1984) for fitting the U.S. income distribution in 1980. For the second sample, we follow Davidson and Flachaire (2007) and set $\mathrm{SM}(0.4, 2.8, 1.7)$ in order to mimic the net income distribution of German households.

We assume a Gaussian copula to capture the sample dependence for matched pairs.

$$C(u_1,u_2;\rho) = \Phi_\rho\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right), \quad (u_1,u_2) \in [0,1]\times[0,1], \quad \rho \in [-1,1], \tag{5.3}$$

where Φ and Φ_ρ denote the cumulative distribution functions (CDFs) of the standard univariate and bivariate normal distributions, respectively. The parameter ρ indicates the degree of sample dependence, with 0 for independence and 1 (-1) for positive (negative) linear dependence.

We will explore various aspects of how sample dependence affects asymptotic confidence intervals (CIs). Recall that it is challenging to predict the value or sign of σ_{12} in (4.15) for a given $\mathrm{Cov}(X_1,X_2)$. This difficulty arises because the functional $\psi(x)$ in (2.1) is often not monotonic in x. To address this issue, we will first evaluate the correlation between $\psi_1(X_{1i})$ and $\psi_2(X_{2i})$, namely ρ_θ in (5.1), against $\rho \in \{0, \pm 0.01, \ldots, \pm 1\}$ in the Gaussian copula (5.3). For simplicity, we consider matched pairs of observations (i.e., $\lambda = 1$). Since the marginal distributions are known, we can directly compute ψ_k for $k = 1,2$.

Figure 1 plots the relationship between ρ_θ and ρ based on 5000 simulations, each consisting of 5000 observations. [3] We find that both curves of the Gini index (left) and LC ordinate (right) are U-shaped but asymmetric at zero, and that ρ_θ is nonnegative for most parameter values considered. Therefore, by considering sample dependence, we anticipate that the proposed inference method using overlapping samples will yield more efficient results due to its smaller asymptotic variance. However, the correlation ρ_θ can be monotonic in ρ. An example is the population mean, where $\psi(x) = x - \mu$ and, thus, $\rho_\theta = \rho$. We then expect that the conventional inference approach, which

[3] For the remaining simulation experiments, especially those involving bootstrap methods, we used 1000 replications. We use the larger number of replications for two reasons. First, we approximate correlation coefficients based on the *true* but unknown influence functions via simulation. The more replications we have, the better the approximation. Second, the computational cost here is much lower than for the experiments in other sections, since bootstrapping is not used.

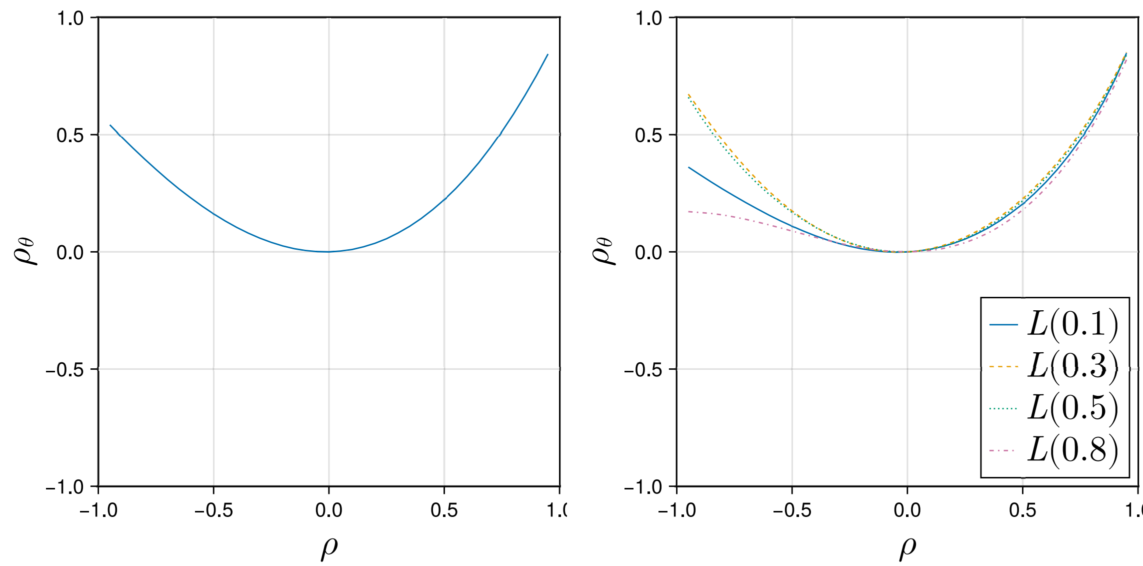

Figure 1. ρ_θ in (5.1) for Gini index (left panel) and LC ordinates (right panel) against ρ of Gaussian copula in (5.3).

NOTE: ρ_θ in (5.1) for the Gini index (left panel) and Lorenz-curve ordinates (right panel) against ρ of the Gaussian copula in (5.3). The data are generated from $F_1 = \mathrm{SM}(1, 1.6971, 8.3679)$ and $F_2 = \mathrm{SM}(0.4, 2.8, 1.7)$ using 5,000 replications of 5,000 observations, $\rho \in \{-0.95, -0.9, \ldots, 0.95\}$, and overlap portion $\lambda = 1$.

ignores sample dependence, will underestimate the asymptotic variance and thus lead to incorrect conclusions.

To quantify the effects of sample dependence, we take the asymptotic standard deviation with independent samples as the benchmark, denoted by σ_Δ^I, where

$$\sigma_\Delta^I = \left(0.5\sigma_1^2 + 0.5\sigma_2^2\right)^{1/2}. \tag{5.4}$$

If we incorrectly specify $\lambda = 0$, we end up with the asymptotic variance σ_Δ^I. We examine the increase in standard deviations using the formula $(\sigma_\Delta^I - \sigma_\Delta^D)/\sigma_\Delta^D$, where σ_Δ^D is in (5.1). Note that σ_Δ^D is affected by the correlation between ψ_1 and ψ_2 (*i.e.*, ρ_θ) and overlap proportion (*i.e.*, λ).

We consider again the marginal distributions $\mathrm{SM}(1, 1.6971, 8.3679)$ and $\mathrm{SM}(0.4, 2.8, 1.7)$. The sample dependence of the matched pairs is described by the Gaussian copula (5.3) with $\rho \in \{0, \pm 0.1, \pm 0.5, \pm 0.99\}$. The overlap portion of matched pairs is $\lambda \in \{0.1, 0.5, 0.9\}$.

Table 2 presents the results based on 5000 replications with 5000 observations each. We use the true function ψ_k for $k = 1, 2$, though typically unknown in practice. Note that the sample dependence can substantially affect the asymptotic standard deviation if samples are strongly correlated or share significant overlap. Take the change in population mean $(\Delta\mu)$ as an example. We find that the increase (decrease) in standard deviations resulting from neglecting sample dependence can be as much as 180.9% (23.6%).

We next examine the performance of 95% confidence intervals (CIs) by different approaches based on 1000 replications of 1000 observations under the same DGP. Our goal is to investigate the impact of sample dependence on the performance of inference. We set the conventional method as the benchmark, which incorrectly assumes that two samples are independent. By (4.26), we estimate the asymptotic standard deviation as follows.

$$\hat{\sigma}_{\Delta}^{D} = [0.5\hat{\sigma}_1^2 + 0.5\hat{\sigma}_2^2 - (m/n)\hat{\rho}_{\theta}\hat{\sigma}_1\hat{\sigma}_2]^{1/2}, \tag{5.5}$$

$$\hat{\sigma}_{\Delta}^{I} = [0.5\hat{\sigma}_1^2 + 0.5\hat{\sigma}_2^2]^{1/2}, \tag{5.6}$$

where for $k = 1,2$, $\hat{\sigma}_k^2$ is the sample variance of $\hat{\psi}_{n_k}(X_{ki})$ in (4.21), and $\hat{\rho}_{\theta}$ is the sample correlation coefficient of $\{(\hat{\psi}_{n_1}(X_{1i}), \hat{\psi}_{n_2}(X_{2i}))\}_{i=1}^{m}$.

In Table 3, we present the simulation results obtained with $\rho = -0.99$ [in (5.3)] for: the differences in mean, Gini index, and Lorenz curve at percentile 0.5. Results with other values of ρ are provided in the Appendix. First, we find that conventional CIs, which fail to account for sample dependence, may suffer from two problems. They can either be overly conservative with excessive widths or they have coverage rates much lower than the nominal level 95% [see columns $(9)-(12)$]. Such findings are consistent with our expectations in Figure 1 and Table 2.

Second, asymptotic and bootstrap CIs for overlapping samples with 999 replications can effectively control the level with reasonable widths [see columns $(5)-(8)$].

Third, we find that the intersection methods (IMs) are valid, though conservative, for all three index differences, as indicated in $(1)-(4)$. Note that the IMs are applicable in more general scenarios and become the only valid procedure when the overlapping-sample assumptions fail to hold; see Section 5.3 for more details.

Table 2. Increases in standard errors for estimated index differences if ignoring sample dependence

ρ	λ	Increases in standard deviations (%)				
		ΔI	$\Delta\mu$	$\Delta L(0.1)$	$\Delta L(0.5)$	$\Delta L(0.8)$
-0.99	0.1	3.1	-3.7	2.0	3.9	0.9
	0.5	18.8	-15.4	11.3	25.3	4.6
	0.9	45.0	-23.6	23.7	70.1	8.8
-0.5	0.1	0.8	-2.0	0.6	0.9	0.5
	0.5	4.3	-9.1	2.8	4.5	2.3
	0.9	8.2	-14.8	5.3	8.5	4.3
-0.1	0.1	0.0	-0.4	0.0	0.0	0.0
	0.5	0.1	-2.1	0.0	0.1	0.1
	0.9	0.2	-3.7	0.1	0.2	0.2
0	0.1	0.0	0.0	0.0	0.0	0.0
	0.5	0.0	0.0	0.0	0.0	0.0
	0.9	0.0	0.0	0.0	0.0	0.0
0.1	0.1	0.1	0.5	0.1	0.1	0.0
	0.5	0.3	2.3	0.3	0.3	0.2
	0.9	0.5	4.3	0.5	0.5	0.3
0.5	0.1	1.1	2.4	1.0	1.1	0.9
	0.5	6.1	14.2	5.5	5.9	4.8
	0.9	11.8	31.2	10.6	11.5	9.2
0.99	0.1	5.0	5.2	5.1	5.0	4.9
	0.5	36.2	39.4	37.8	36.3	35.7
	0.9	142.0	180.9	159.6	143.7	137.8

Note – The table provides the percentage increases in standard deviations for estimators of several welfare index differences when ignoring sample dependence, including the changes in Gini indices (ΔI), population mean ($\Delta\mu$) and LC ordinates ($\Delta L(p)$) at $p \in \{0.1, 0.5, 0.8\}$. The percentage increases in standard deviations are measured by $(\sigma^I_\Delta - \sigma^D_\Delta)/\sigma^D_\Delta$, where σ^I_Δ and σ^D_Δ are given in (5.4) and (5.1), respectively. The results are based on 5000 replications of 5000 observations according to the DGP: $F_1 = \mathrm{SM}(1, 1.6971, 8.3679)$, $F_2 = \mathrm{SM}(0.4, 2.8, 1.7)$, $\rho \in \{0, \pm 0.5, \pm 0.99\}$ in (5.3), overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$.

Table 3. Performance of proposed and conventional 95% confidence intervals for differences in the mean, Gini index and Lorenz curve at 0.5

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
		Intersection method				Overlapping samples				Independent samples (incorrect)			
$\Delta\theta$	λ	Coverage (%)		Width		Coverage (%)		Width		Coverage (%)		Width	
		Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot
$\Delta\mu$	0.1	99.7	99.7	0.0534	0.0535	94.9	94.6	0.0344	0.0345	94.3	93.7	0.0331	0.0332
	0.5	99.5	99.6	0.0534	0.0535	94.8	94.6	0.0391	0.0392	89.3	90.1	0.0331	0.0332
	0.9	98	97.6	0.0534	0.0537	94.4	94	0.0433	0.0435	**84.3**	**83.2**	0.0331	0.0332
ΔI	0.1	99.9	99.8	0.0664	0.0673	93.8	93.5	0.0399	0.0403	94.6	95.2	0.0411	0.0415
	0.5	100	100	0.0665	0.0676	94.7	94.1	0.0346	0.0351	98.1	98.2	0.0412	0.0416
	0.9	100	100	0.0665	0.0676	95.6	94.6	0.0284	0.0286	**99.7**	**99.8**	0.0412	0.0416
$\Delta L(0.5)$	0.1	99.7	99.6	0.045	0.0458	94.9	95.2	0.0268	0.0273	95	95.4	0.0279	0.0284
	0.5	100	100	0.045	0.0459	94.9	95.3	0.0223	0.0228	98.5	98.5	0.0279	0.0284
	0.9	100	100	0.0451	0.0459	95.7	96.1	0.0164	0.017	**100**	**99.9**	0.0279	0.0284

Note – The table provides coverage rates and widths of 95% confidence intervals for the differences in mean, Gini index and Lorenz curve at percentile 0.5. The data are generated according to DGP: $F_1 = \text{SM}(1, 1.6971, 8.3679)$, $F_2 = \text{SM}(0.4, 2.8, 1.7)$, $\rho = -0.99$ in (5.3), overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$. The sample sizes are both 1000. Columns (1) - (4) are performance by asymptotic intersection method (Asym), bootstrap intersection method (Boot) in section 3. Columns (5) - (8) are performance by asymptotic (Asym) and bootstrap (Boot) methods with overlapping samples in Section 4, using $\hat{\sigma}^D_\Delta$ in (5.5). Columns (9) - (12) are performance by asymptotic (Asym) and bootstrap (Boot) methods if incorrectly ignoring the sample dependence, using $\hat{\sigma}^I_\Delta$ in (5.6). The results are based on 1000 replications with 999 bootstrap repetitions each. The bold font indicates cases where coverage rates largely deviate from 95%.

5.2. Effects of heavy tails

Income distributions often exhibit heavy tails which are notorious for causing problems for both asymptotic and bootstrap inference. One theoretical reason is the need, emphasized by Bahadur and Savage (1956), for restrictions beyond mere existence of the mean. We examine in this section the impacts of heavy tails on the performance of the proposed inference methods. We still consider the SM distribution $\mathrm{SM}(b,a,q)$ in (5.2). The upper tail behaves similarly to the Pareto distribution with parameter $\beta = aq$, also known as the stability index. A smaller β indicates a heavier upper tail. For a variance to exist, β must be larger than two.

To mimic the U.S. income distribution in 1980, we set the distribution of the first sample to $\mathrm{SM}(1, 1.6971, 8.3679)$. For the second sample, we explore three SM distributions, $\mathrm{SM}(0.4, 2.8, 1.7)$, $\mathrm{SM}(0.4, 2, 1.5)$, and $\mathrm{SM}(0.4, 1.4, 1.5)$. The corresponding stability indices are 4.76, 3, and 2.1, indicating increasingly heavier upper tails.

For simplicity, we assume that the two sample sizes are identical with $n_1 = n_2 = n \in \{100, 200, 500, 1000\}$. A portion $\lambda \in \{0.1, 0.5, 0.9\}$ of observations is drawn in pairs, with dependence characterized by a Gaussian copula in (5.3) with $\rho = -0.99$. We provide in the Appendix additional results for other values of ρ. The performance of CIs is evaluated based on 1000 replications, with $B = 999$ bootstrap repetitions each.

The DGPs considered are the following ones.

DGP I-A: $F_1 = \mathrm{SM}(1, 1.6971, 8.3679)$, $F_2 = \mathrm{SM}(0.4, 2.8, 1.7)$, $\rho = -0.99$ in (5.3), stability index of F_2 is $\beta_2 = 4.76$, overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$.

DGP I-B: $F_1 = \mathrm{SM}(1, 1.6971, 8.3679)$, $F_2 = \mathrm{SM}(0.4, 2, 1.5)$, $\rho = -0.99$ in (5.3), stability index of F_2 is $\beta_2 = 3$, overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$.

DGP I-C: $F_1 = \mathrm{SM}(1, 1.6971, 8.3679)$, $F_2 = \mathrm{SM}(0.4, 1.4, 1.5)$, $\rho = -0.99$ in (5.3), stability index of F_2 is $\beta_2 = 2.1$, overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$.

As indicated by Figure 1, we expect similar performance for the Gini index and Lorenz curve differences. To save space, we present results for the Gini index and population mean, and provide the results for the difference in LC ordinate at the percentile of 0.5 in the Appendix. We assess the performance of four proposed confidence intervals (CIs) at the 95% level. These include the asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic inference with overlapping samples (OS-Asym), and bootstrap inference with overlapping samples (OS-Boot).

We also include the robust-t method (denoted by t) proposed by Ibragimov et al. (2025) as a comparison, which is designed to handle heavy-tailed distributions.[4] Before presenting simulated results, we highlight several differences in methodology between our approaches and the robust-t method. First, the robust-t method does not, in theory, apply to overlapping samples, let alone to generic dependent samples, unless the two samples are independent or matched pairs. Second, the robust-t method assumes that, for matched pairs, the two estimators in each group converge *jointly* to a bivariate Gaussian distribution. By contrast, the intersection method (IM-Asym and IM-Boot) only needs estimators in each sample to converge weakly to a univariate Gaussian distribution; see

[4] We thank the anonymous referee for giving this suggestion.

Proposition 3.1 and 3.2. This generality allows for arbitrary dependence between the two samples, and estimators which may *not* be jointly weakly convergent. Third, the choice of the number of groups, say q, is critical for the performance of the robust-t method. Here, q is fixed, often small, and cannot be optimized using data without further assumptions. The tuning parameter q should be large under 'unpleasant situations' such as heavy tails, distinct distributions. However, according to Ibragimov and Müller (2010, Theorem 1), the usual two-sided t-test of level 10% (or more) is *not* automatically conservative for large q.

For DGP I-A, we illustrate the performance of CIs for the difference in mean and Gini index in Table 4. As expected, the asymptotic and bootstrap CIs with overlapping samples (OS-Asym and OS-Boot) exhibit coverage rates close to the nominal level of 95% due to the absence of heavy tails. However, the asymptotic and bootstrap intersection methods (IM-Asym and IM-Boot) are conservative, with consistently higher coverage rates than the nominal level, yielding a trade-off between efficiency and robustness. The robust-t method is less reliable in these designs due to its insufficient coverage rates; see the results for $\Delta\mu$ when $\lambda = 0.9$. Moreover, the robust-t method has a wider CI, and is thus less efficient than methods based on overlapping samples (OS-Asym and OS-Boot). We also show in the appendix that the robust-t approach has a much lower coverage at lower nominal levels such as 90% and 85%.

For DGP I-B, we present the performance of CIs in Table 5. In this scenario, the second distribution has a heavy upper tail. The coverage rates of asymptotic CIs with overlapping samples are lower than 95% when two samples are strongly positively related; see, for example, column OS-Asym when $\rho = 0.99$ and $\lambda = 0.9$. The bootstrap CI (OS-Boot), however, can alleviate this issue. The asymptotic and bootstrap IMs (IM-Asym and IM-Boot) are still conservative. Nevertheless, the widths are considered acceptable compared to the bootstrap CIs. The robust-t method is less competitive than IM approaches, as it has lower coverage but exhibits CI widths that are similar or even larger.

For DGP I-C, we evaluate the performance of CIs for the two indices differences in Table 6. The second distribution now has an extremely heavy tail, leading to considerable variance. Neither the asymptotic nor the bootstrap CIs with overlapping samples (OS-Asym and OS-Boot) yield reliable results, as their coverage falls below 95%. However, the bootstrap CI (OS-Boot) suffers less coverage distortion compared to the asymptotic CI. The IMs (IM-Asym and IM-Boot) remain valid and provide reasonable widths, sometimes comparable to those of OS-Boot. We confirm previous findings under DGP I-B for the robust-t method. Moreover, the robust-t method becomes less reliable than IMs due to its insufficient coverage rates; see the results for $\Delta\mu$ when $\lambda = 0.9$.

The simulations yield four main findings. First, the asymptotic CIs based on overlapping samples (OS-Asym) perform quite well, with coverage close to the nominal level and a narrow width, except in the case of heavy tails. Second, bootstrap inference with overlapping samples (OS-Boot) is effective for heavy-tailed distributions, as long as the tails are not too heavy. Such a finding is consistent with the literature; see, for example, Davidson (2009, 2012). Third, the intersection methods (IMs) tend to be conservative, unless the tails are so heavy that the variance does not exist; see a numerical exercise in the appendix. Finally, the robust-t method is less efficient than the OS-based method for non-heavy-tailed distributions, and less reliable than IMs for heavy-tailed distributions.

Table 4. DGP I-A: coverage and width of 95% confidence intervals for mean and Gini index differences

$\Delta\theta$	λ	n	Coverage					Width				
			IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
$\Delta\mu$	0.1	100	99.8	99.8	94.8	93.9	97.6	0.1661	0.1724	0.1077	0.1081	0.1607
		200	99.7	99.8	94.6	94	98.3	0.1188	0.1213	0.0768	0.0768	0.1151
		500	99.6	99.6	94	93.5	97.8	0.0754	0.076	0.0486	0.0487	0.0732
		1000	99.7	99.7	96.6	95.6	97.7	0.0536	0.0538	0.0345	0.0346	0.0514
	0.5	100	99.3	99.3	94.9	94.2	96.1	0.167	0.174	0.1232	0.1239	0.1583
		200	99.3	99.3	95.6	95.1	95.6	0.1184	0.121	0.087	0.0873	0.1129
		500	98.9	98.9	95	94.7	96.2	0.0752	0.0758	0.0552	0.0554	0.071
		1000	99	99	94.9	94.8	96	0.0534	0.0536	0.0391	0.0392	0.0517
	0.9	100	97.9	98	94.2	93.8	93.8	0.1668	0.1738	0.1361	0.1381	0.1611
		200	98.1	98.1	94.4	94.3	94.3	0.1188	0.1215	0.0966	0.0974	0.1154
		500	98.1	97.9	94.5	94.2	93.2	0.0756	0.0762	0.0614	0.0618	0.0725
		1000	97.9	97.9	93.9	94.1	91.9	0.0535	0.0538	0.0434	0.0436	0.0499
ΔI	0.1	100	99.6	99.9	94.9	94.5	98.1	0.2011	0.2208	0.122	0.1265	0.2023
		200	99.9	99.9	94.3	94.2	98.7	0.1452	0.1534	0.0877	0.0899	0.1428
		500	99.9	99.8	95.9	95.7	98.1	0.0933	0.0957	0.0561	0.057	0.0902
		1000	100	99.9	95.4	94.5	98.3	0.0667	0.0677	0.0401	0.0404	0.0649
	0.5	100	100	100	92.8	92.2	99.3	0.2029	0.2247	0.1064	0.1103	0.2053
		200	99.9	99.9	94.7	94	98.8	0.1448	0.153	0.0755	0.0772	0.1414
		500	100	100	95.1	95.1	99.3	0.093	0.0954	0.0483	0.049	0.0892
		1000	100	100	95.5	95.1	99.4	0.0665	0.0675	0.0346	0.0349	0.0644
	0.9	100	100	100	94.7	93.9	99.5	0.2018	0.2228	0.086	0.0889	0.2025
		200	100	100	94.1	93	99.2	0.1461	0.1547	0.062	0.0633	0.1465
		500	100	100	95	94	99.4	0.0939	0.0967	0.0401	0.0408	0.089
		1000	100	100	95.4	95.3	99.4	0.0668	0.068	0.0285	0.029	0.0634

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean difference ($\Delta\mu$) and Gini index differences (ΔI) by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-A with overlap portion λ and sample size n. The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 5. DGP I-B: coverage and width of 95% confidence intervals for mean and Gini index differences

$\Delta\theta$	λ	n	Coverage					Width				
			IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
$\Delta\mu$	0.1	100	99.3	99.4	93.1	92.2	96.2	0.2448	0.2753	0.1668	0.1788	0.2494
		200	99.1	99.4	93.8	93.7	97.3	0.1753	0.1933	0.1192	0.1282	0.1797
		500	99.4	99.6	93.8	94.3	96.9	0.1114	0.1158	0.0754	0.0774	0.115
		1000	99.3	99.2	93.5	93.3	96.1	0.0798	0.0816	0.0541	0.055	0.0815
	0.5	100	98.2	98.7	93.6	92.8	95.5	0.2411	0.2669	0.1796	0.1878	0.2455
		200	98.6	99	94	93.1	95.8	0.1755	0.1879	0.1304	0.1353	0.1767
		500	99.1	99.1	94.4	94.1	96.4	0.1137	0.1201	0.0843	0.0876	0.1156
		1000	98.9	98.7	94.6	94.3	94.7	0.0797	0.0815	0.0588	0.0597	0.0809
	0.9	100	98.5	98.2	94.1	94.3	94.4	0.2454	0.2729	0.197	0.2065	0.2465
		200	98.3	98.5	95	94.5	94.7	0.176	0.1893	0.1407	0.1461	0.1812
		500	98.1	98.1	95.2	94.5	94	0.1129	0.1176	0.0899	0.092	0.1133
		1000	99.1	98.9	95.4	95.1	95.2	0.0809	0.0838	0.0642	0.0657	0.0833
ΔI	0.1	100	99.5	99.5	92.3	93.1	98.5	0.2506	0.301	0.1569	0.1808	0.2557
		200	99.4	99.6	91.7	92.3	97.7	0.1847	0.2113	0.1154	0.1297	0.1848
		500	99.4	99.6	93.6	93.6	97.6	0.1208	0.1306	0.0753	0.0806	0.1213
		1000	99.6	99.5	94.6	94.1	98.5	0.0876	0.0927	0.0548	0.0577	0.0866
	0.5	100	99.7	99.9	92	92.6	98.8	0.2469	0.2937	0.1376	0.1559	0.2489
		200	99.9	99.9	93.8	93.8	98.7	0.1848	0.2106	0.104	0.1165	0.1861
		500	99.8	99.9	93.4	92.6	98.8	0.1231	0.1351	0.0703	0.0782	0.1235
		1000	99.9	100	94.3	93.9	98.4	0.0875	0.0925	0.05	0.0531	0.0871
	0.9	100	99.9	99.9	91.1	91.3	99.4	0.2509	0.3007	0.1229	0.1445	0.256
		200	100	100	90.5	91.1	98.7	0.1851	0.2121	0.0924	0.1077	0.1844
		500	100	100	92.4	93.4	99	0.1224	0.1341	0.0622	0.0702	0.1216
		1000	100	100	92.8	93	99.2	0.0894	0.0963	0.0464	0.0519	0.0879

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean difference ($\Delta\mu$) and Gini index differences (ΔI) by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-B with overlap portion λ and sample size n. The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 6. DGP I-C: coverage and width of 95% confidence intervals for mean and Gini index differences

$\Delta\theta$	λ	n	Coverage					Width				
			IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
$\Delta\mu$	0.1	100	97.1	97.9	88.2	91.2	94	0.4272	0.717	0.3192	0.5391	0.4919
		200	96.2	97.6	91	91.3	94	0.3162	0.4507	0.2369	0.3361	0.3612
		500	97.2	98.3	90.6	92.6	94.1	0.2173	0.3385	0.1642	0.267	0.2513
		1000	98	98.9	91.9	91.9	94.1	0.1581	0.1975	0.1197	0.1533	0.1815
	0.5	100	95.2	97.6	89.4	91.5	92.2	0.4467	0.8853	0.3513	0.6615	0.5092
		200	96	97.6	91.3	92.7	93.4	0.3251	0.4581	0.2545	0.3455	0.366
		500	97.1	98.3	93.1	94.3	94.4	0.218	0.2843	0.1709	0.2196	0.2534
		1000	98.1	98.4	94.2	94.4	94.2	0.1626	0.2166	0.1277	0.1685	0.188
	0.9	100	95	96.6	90.4	92	91	0.4493	0.9286	0.3659	0.6626	0.5146
		200	94.6	96.7	89.3	91.4	91.3	0.3075	0.4093	0.2483	0.3104	0.3508
		500	97.1	98	93.1	94.2	93.6	0.215	0.2949	0.1738	0.228	0.2469
		1000	97.1	98.3	91.9	92.1	94.1	0.1616	0.2286	0.1305	0.1802	0.1852
ΔI	0.1	100	97.6	98.2	85	89.1	95.2	0.2898	0.374	0.1878	0.2429	0.3063
		200	98.3	99.1	88.1	91	95.3	0.2277	0.2851	0.1496	0.1895	0.2345
		500	98.2	99.1	89.4	92.2	96.3	0.1631	0.1998	0.1094	0.1371	0.1711
		1000	99	99.7	91.5	93.2	96.7	0.1235	0.1485	0.0838	0.1034	0.1285
	0.5	100	98.6	99.2	85.2	89	97.5	0.2941	0.3843	0.1789	0.2386	0.3102
		200	97.9	99.1	84.1	87.2	95.9	0.231	0.2916	0.1434	0.1907	0.2427
		500	98.8	99.5	88.8	92.1	97	0.1648	0.2028	0.1058	0.1378	0.1709
		1000	98.9	99	88.4	91.4	96.3	0.1249	0.1493	0.0816	0.1044	0.1311
	0.9	100	98.8	99.3	82.7	86.9	95.5	0.2958	0.389	0.1663	0.2298	0.3115
		200	98.8	99.2	84.3	88.6	95.9	0.2268	0.2863	0.131	0.175	0.2332
		500	99.2	99.4	88.7	92.2	96.9	0.1611	0.1949	0.0976	0.1294	0.1641
		1000	98.8	99.3	89	91.7	97	0.1232	0.1474	0.0768	0.0995	0.1296

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean difference ($\Delta\mu$) and Gini index differences (ΔI) by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-C with overlap portion λ and sample size n. The numbers are based on 1000 replications with 999 bootstrap repetitions each.

5.3. Beyond overlapping samples

This section examines scenarios where the overlapping-sample framework in Assumption 4.1 may not be applicable. We take the mean difference $\Delta\mu = \mu_1 - \mu_2$ as an example for two reasons. First, ψ in (2.1) has a simple expression of $\psi(x) = x - \mu$. As $\psi(x)$ is increasing in x, we can compute ρ_θ in (4.25) as $\rho_\theta = \rho$. Second, the population moments are usually easier to calculate than the welfare indices.

We demonstrate the robustness of intersection methods (IM) by comparing them to approaches based on overlapping samples. Specifically, we aim to investigate the performance of various CIs when Assumption 4.1 fails, while Assumption 3.1 remains valid. We also include the robust-*t* method (t) and a modified version using random split (t-Shuffle) for comparison.

First, we consider samples that violate Assumption 4.1 (**A2**). Suppose that pairs of unobserved observations, denoted by $\{(U_{1i}, U_{2i})\}_{i=1}^n$ are iid data from the joint distribution (F_1, F_2, ρ), where $F_1 = \text{SM}(1, 1.6971, 8.3679)$ and $F_2 = \text{SM}(0.4, 1.4, 1.5)$. The correlation coefficient ρ in (5.3) is -0.95. Suppose that we only observe the ordered values of $\{(U_{1i}, U_{2i})\}_{i=1}^n$. Let $X_{k,i} = U_{k(i)}$ for $k = 1, 2$ and $U_{k(1)} \leq \cdots, \leq U_{k(n)}$. The overlap of samples are then $(X_{1i}, X_{2i})_{i=1}^n$. The condition (**A2**) is violated in this case. The unmatched observations are iid data from F_1 and F_2. The sample sizes are $n = 100, 200, 500, 1000$ with overlap portions $\lambda = 0.5, 1$. So, the DGP is as follows.

DGP II-A: (Violate (A2)) $\{X_{k,i}\}_{i=1}^n = \{U_{k(i)}\}_{i=1}^n$ for $k = 1, 2$, where $U_{k(i)}$ is the i-th order statistic. $\{U_{1,i}\}_{i=1}^n \overset{iid}{\sim} \text{SM}(1, 1.6971, 8.3679)$, $\{U_{2,i}\}_{i=1}^n \overset{iid}{\sim} \text{SM}(0.4, 1.4, 1.5)$, $\rho = -0.95$. The sample sizes are $n = 100, 200, 500, 1000$. The overlap portions are $\lambda = 0.5, 1$.

Second, we conduct an experiment where condition Assumption (**A3**) fails. We assume that the second sample consists of $2n$ iid observations from $\text{SM}(0.4, 1.4, 1.5)$. Given the sample $\{X_{2,i}\}_{i=1}^{2n}$, define $X_{1,i} = X_{2,2i-1} - X_{2,2i}$ for $i = 1, \ldots, n$. Let $\{(X_{1,i}, X_{2,2i-1})\}_{i=1}^n$ be the matched pairs. By construction, $\{X_{1,i}\}_{i=1}^n$ and $\{X_{2,2i-1}\}_{i=1}^n$ are dependent, which violates condition (**A3**). The DGP is summarized as follows.

DGP II-B: (Violate (A3)) $\{X_{2,i}\}_{i=1}^{2n} \overset{iid}{\sim} \text{SM}(0.4, 1.4, 1.5)$ and $\{X_{1,i}\}_{i=1}^n$ where $X_{1,i} = X_{2,2i-1} - X_{2,2i}$. $n = 100, 200, 500, 1000$.

Table 7 provides simulation results under DGP II-A and DGP II-B where Assumption 4.1 fails. The results demonstrate that intersection methods (IMs) are effective, but overlapping-sample (OS) methods are invalid. The robust-*t* method is not valid due to its inadequate coverage. In contrast, the robust-*t* method with random splits is less efficient than IMs, with wider CIs and lower coverage rates. We present additional results in the appendix for alternative parameter values.

Table 7. Coverage and width of 95% confidence intervals for mean differences for samples that violate the overlapping-sample assumptions

DGP	λ	n	Coverage						Width					
			IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	t-Shuffle	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	t-Shuffle
II-A	0.5	100	94.9	97	82.8	86.2	100	93	0.4265	0.6491	0.2826	0.483	1.0731	0.4815
		200	96.6	98.1	85	88.6	100	93.5	0.3206	0.4797	0.2158	0.3714	1.0396	0.3702
		500	97.3	98.3	88.1	90.1	100	94.1	0.2085	0.2531	0.1415	0.1845	0.9979	0.2398
		1000	96.7	97.4	86.5	88.1	100	94.3	0.1556	0.1965	0.1073	0.1489	0.9919	0.1763
	1	100	95.1	97.1	74.5	79.6	100	92.4	0.4341	0.8141	0.2527	0.8615	1.8434	0.4984
		200	95.2	97.4	77.5	80.8	100	93.2	0.3124	0.4055	0.1846	0.3044	1.8257	0.354
		500	96.3	97.2	78.7	81.2	100	93.9	0.2112	0.2609	0.1295	0.1919	1.8442	0.2452
		1000	97.4	97.5	80.8	82.9	100	94.2	0.165	0.2421	0.1061	0.2049	1.8531	0.1927
II-B	-	100	99.4	99.1	85.8	71.9	97.1	97.7	0.7913	1.0319	0.4033	0.5179	0.8207	0.8043
		200	99.7	99.5	85.9	71.9	97.6	97.4	0.5735	0.6662	0.2941	0.3521	0.5882	0.5868
		500	99.3	99	88.1	73.7	98.5	96.5	0.3911	0.4901	0.2006	0.2652	0.3972	0.396
		1000	99.6	99.3	86.7	72.4	97.7	98.2	0.2909	0.3359	0.1477	0.1732	0.2944	0.2984

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean difference by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot), robust-t method with four groups (t) and robust-t method using the random split with four groups (t-Shuffle). The data are generated according to DGP II-A and II-B with overlap portion λ (if applicable) and sample size n. The numbers are based on 1000 replications with 999 bootstrap repetitions each.

6. Application to the SHIW Data of the Bank of Italy

This section examines the changes in household financial inequality between 2012 and 2014 in Italy. We utilize the data from the Survey of Household Income and Wealth (SHIW) conducted by the Bank of Italy. These data have been used by several studies, including Raffinetti, Siletti and Vernizzi (2015), Raffinetti, Siletti and Vernizzi (2017), Bouezmarni and Dufour (2024) and Dufour and He (2024*b*).

For simplicity, apart from the modeled overlap across waves, we treat observations within each wave as iid. To address the heterogeneity in household structures, we normalize data by the equivalence scale proposed by Kakwani and Lambert (1998):

$$hhscale_i = (ad_i + 0.2ch_{1,i} + 0.4ch_{2,i} + 0.7ch_{3,i})^{0.8} + 0.1w_i \tag{6.1}$$

where, for family i, ad_i is the number of adults within a family, $\{ch_{j,i}\}_{j=1}^{3}$ denote the number of children with age in $(0,5]$, $[6,14]$ and $[15,17]$, respectively, and w_i is the number of employees or self-employed individuals in household i. Table 8 presents some summary statistics, and Figure 2 plots the estimated densities and distribution functions.

To analyze the changes in financial inequality in Italy from 2012 to 2014, we must address two key issues. First, there is an overlap in the samples, as nearly half of the observations appear in both waves. Therefore, we must take the sample dependence into account. Second, financial income can be negative, indicating that some households experienced losses. According to Table 8, 10% of households suffered losses in 2012, while 8.5% faced losses in 2014, respectively. We therefore consider a family of extended inequality measures proposed by Bouezmarni and Dufour (2024) for possibly negative variables. For convenience, we focus on two specific types of measures.

Definition 6.1 EXTENDED LORENZ CURVE. *Consider a general variable X with distribution F. Let $\mu(F) = \int x\, dF$ be the mean, $Q(p;F) = \inf\{x : F(x) \geq p\}$ be the p-th quantile, and $\mu^p(F) = \int |x|\, dF$ be the absolute mean.*

(1) *Let $F^{\oplus}(x) = \Pr(X \leq x \mid X > 0)$ and $F^{\ominus}(x) = \Pr(X \leq x \mid X < 0)$. Then the sign-conditional Lorenz curves are*

$$L^{\oplus}(p;F^{\oplus}) = \frac{\int_0^p Q(u;F^{\oplus})\, du}{\mu(F^{\oplus})}, \quad L^{\ominus}(p;F^{\ominus}) = \frac{\int_0^p Q(u;F^{\ominus})\, du}{\mu(F^{\ominus})}. \tag{6.2}$$

(2) *The signed Lorenz curve is*

$$L^{s}(p;F) = \frac{\int_0^p Q(u;F)\, du}{\mu^p(F)}. \tag{6.3}$$

(3) *Let $X_1^{\oplus}$ and $X_2^{\oplus}$ (resp. $X_1^{\ominus}$ and $X_2^{\ominus}$) be independent copies of variable $X^{\oplus}$ (resp. $X^{\ominus}$) with CDF $F^{\oplus}$ (resp. $F^{\ominus}$). Then the sign-conditional Gini indices are*

$$I^{\oplus}(F^{\oplus}) = \frac{\mathbb{E}(|X_1^{\oplus} - X_2^{\oplus}|)}{2\mu(F^{\oplus})}, \quad I^{\ominus}(F^{\ominus}) = \frac{\mathbb{E}(|X_1^{\ominus} - X_2^{\ominus}|)}{2|\mu(F^{\ominus})|}. \tag{6.4}$$

Table 8
Descriptive statistics for household financial income.

Statistic	2012	2014
n	8151	8156
m	4459	4459
$\Pr(X<0)$	0.1002	0.0858
$\Pr(X=0)$	0.187	0.1629
$\Pr(X>0)$	0.7128	0.7512
Min	-17123.2	-10067
$Q(0.25)$	0	0.3183
Mean	189.5418	108.1888
Median	29.6702	24.8268
$Q(0.75)$	154.7083	125.8219
Max	73127.16	24280.84
Skewness	15.703	7.6478
Kurtosis	528.6745	157.134
$\mathbb{E}(X\|X>0)$	458.921	285.0303
$\mathbb{E}(X\|X<0)$	-1372.55	-1234.28
$\mathbb{E}(\|X\|)$	464.6487	320.0156

Note— n is sample size. m is the size of overlap. $Q(p)$ is the p-th quantile.

(4) *Let X_1 and X_2 be independent copies of variable X with CDF F. The positive Gini index is*

$$I^p(F) = \frac{\mathbb{E}(|X_1 - X_2|)}{2\mu^p(F)}. \tag{6.5}$$

The sign-conditional LCs and Gini indices are tools for analyzing internal inequality levels inside the group of winners or losers. The signed LC and positive Gini index measure the overall level of inequality. See Bouezmarni and Dufour (2024) and Dufour and He (2024*b*) for more discussion. To simplify our discussion, we call $L^{\oplus}$ (resp. $L^{\ominus}$) the positive (resp. negative) truncated LC and analogously to $I^{\oplus}$ and $I^{\ominus}$.

To facilitate our discussion, we use the superscript † (or $*$) to simplify notations to indicate a specific object from a collection. For example, $(L^{\dagger}, F^{\dagger}) \in \{(L^{\oplus}, F^{\oplus}), (L^{\ominus}, F^{\ominus})\}$ means that $(L^{\dagger}, F^{\dagger})$ can be either of the sign-conditional LCs with their corresponding distributions. We will omit the collection if no confusion arises.

We estimate the extended inequality measures by plugging the associated EDFs. The estimated extended LCs are plotted in the left panels of Figure 3, 4, and 5. The estimates for extended Gini indices are displayed in Table 9. We will then conduct statistical inference for these estimates. As shown in Dufour and He (2024*b*), the estimators have ALG representations under specific conditions (outlined below).

Lemma 6.1 INFLUENCE FUNCTIONS OF ESTIMATED EXTENDED INEQUALITY MEASURES.

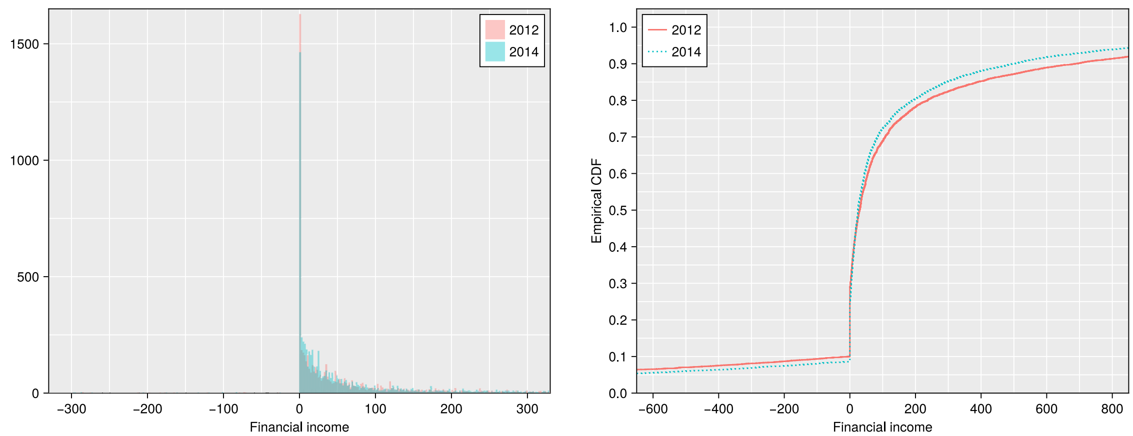

Figure 2. 2014 vs 2012: distributions of financial income.

Suppose $\{X_i\}_{i=1}^n \overset{iid}{\sim} F$ *where distribution function* F *has a finite variance.*

(1) *(Extended LC ordinate) Let* $(L^\dagger, F^\dagger) \in \{(L^\oplus, F^\oplus), (L^\ominus, F^\ominus), (L^s, F)\}$. *If* $F^\dagger$ *is differentiable at its p-th quantile, then* $\hat{L}^\dagger(p)$ *satisfies Definition 2.1 with*

$$\psi^\dagger(x; L^\dagger, F^\dagger, p) = W^\dagger(x; L^\dagger, F^\dagger, p) - \mathbb{E}[W^\dagger(X^\dagger; L^\dagger, F^\dagger, p)] \tag{6.6}$$

where $(W^\dagger, X^\dagger) \in \{(W^\oplus, X^\oplus), (W^\ominus, X^\ominus), (W^s, X)\}$

$$\begin{aligned} W^\oplus(x; L^\oplus, F^\oplus, p) &= \frac{\left[x - Q(p; F^\oplus)\right] \mathbf{1}\left(x \le Q(p; F^\oplus)\right) - x L^\oplus(p; F^\oplus)}{\mu(F^\oplus)}, \\ W^\ominus(x; L^\ominus, F^\ominus, p) &= \frac{\left[x - Q(p; F^\ominus)\right] \mathbf{1}\left(x \le Q(p; F^\ominus)\right) - x L^\ominus(p; F^\ominus)}{\mu(F^\ominus)}, \\ W^s(x; L^s, F, p) &= \frac{[x - Q(p; F)] \mathbf{1}(x \le Q(p; F)) - |x| L^s(p; F)}{\mu^p(F)}. \end{aligned} \tag{6.7}$$

(2) *(Extended Gini index) Let* $(I^\dagger, F^\dagger) \in \{(I^\oplus, F^\oplus), (I^\ominus, F^\ominus), (I^p, F)\}$. *If* $F^\dagger$ *is continuous, then* $\hat{I}^\dagger$ *satisfies the Definition 2.1 with*

$$\psi^\dagger(x; I^\dagger, F^\dagger) = K^\dagger(x; I^\dagger, F^\dagger) - \mathbb{E}[K^\dagger(X^\dagger; I^\dagger, F^\dagger)] \tag{6.8}$$

where $(K^\dagger, X^\dagger) \in \{(K^\oplus, X^\oplus), (K^\ominus, X^\ominus), (K^p, X)\}$

$$\begin{aligned} K^\oplus(x; I^\oplus, F^\oplus) &= \frac{2xF^\oplus(x) - 2\int^x t\, dF^\oplus(t) - [I^\oplus(F^\oplus)+1]x}{\mu(F^\oplus)}, \\ K^\ominus(x; I^\ominus, F^\ominus) &= \frac{2xF^\ominus(x) - 2\int^x t\, dF^\ominus(t) - [I^\ominus(F^\ominus)+1]x}{\mu(F^\ominus)}, \\ K^p(x; I^p, F) &= \frac{2xF(x) - 2\int^x t\, dF(t) - I^p(F)\,|x| - x}{\mu^p(F)}. \end{aligned} \tag{6.9}$$

Under Assumption 4.1, one can verify that the above estimates fulfill Assumption 4.2 by multivariate central limit theorem. We then obtain the following corollary.

Corollary 6.2 CONVERGENCE IN DISTRIBUTION FOR THE EXTENDED INEQUALITY DIFFERENCE. *Suppose that the Assumption 4.1 holds with finite variance matrix.*

(1) *(Extended LC ordinate) If the distribution* $F_k^\dagger \in \{F_k^\oplus, F_k^\ominus, F_k\}$*,* $k = 1,2$*, is differentiable at its p-th quantile, then*

$$\sqrt{N}\big(\Delta \hat{L}^\dagger(p) - \Delta L^\dagger(p)\big) \xrightarrow{d} \mathrm{N}\big(0, \Sigma_\Delta^\dagger(p)\big) \tag{6.10}$$

where $\Delta L^\dagger \in \{\Delta L^\oplus, \Delta L^\ominus, \Delta L^s\}$*, and* $\Sigma_\Delta^\dagger \in \{\Sigma_\Delta^\oplus, \Sigma_\Delta^\ominus, \Sigma_\Delta^s\}$ *is such that*

$$\begin{aligned} \Sigma_\Delta^\dagger(p) =& (1-\eta_1)\mathrm{Var}\big[W_1^\dagger\big(X_1^\dagger; p\big)\big] + \eta_1 \mathrm{Var}\big[W_2^\dagger\big(X_2^\dagger; p\big)\big] \\ & - 2\sqrt{\eta_1(1-\eta_1)\lambda_1\lambda_2}\mathrm{Cov}\big[W_1^\dagger\big(X_1^\dagger; p\big), W_2^\dagger\big(X_2^\dagger; p\big)\big], \end{aligned} \tag{6.11}$$

with $W_k^\dagger(x;p) = W^\dagger(x; L^\dagger, F_k^\dagger, p)$ *given by* (6.7)*.*

(2) *(Extended Gini index) If the distribution* F_k*,* $k = 1,2$*, is continuous with a possible jump at zero, then*

$$\sqrt{N}\big(\Delta \hat{I}^\dagger - \Delta I^\dagger\big) \xrightarrow{d} \mathrm{N}\big(0, V_\Delta^\dagger\big) \tag{6.12}$$

where $\Delta I^\dagger \in \{\Delta I^\oplus, \Delta I^\ominus, \Delta I^p\}$*, and* $V_\Delta^\dagger \in \{V_\Delta^\oplus, V_\Delta^\ominus, V_\Delta^p\}$ *is such that*

$$\begin{aligned} V_\Delta^\dagger =& (1-\eta_1)\mathrm{Var}\big[K_1^\dagger\big(X_1^\dagger\big)\big] + \eta_1 \mathrm{Var}\big[K_2^\dagger\big(X_2^\dagger\big)\big] \\ & - 2\sqrt{\eta_1(1-\eta_1)\lambda_1\lambda_2}\mathrm{Cov}\big[K_1^\dagger\big(X_1^\dagger\big), K_2^\dagger\big(X_2^\dagger\big)\big] \end{aligned} \tag{6.13}$$

with $K_k^\dagger(x) = K^\dagger(x; I^\dagger, F_k^\dagger)$ *given by* (6.9)*.*

According to (4.26), we estimate the asymptotic variance by the sample variance of the estimated influence function. We then obtain CIs for the inequality changes with confidence level 95% using four proposed methods, including the asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic inference with overlapping samples (OS-Asym), and bootstrap inference with overlapping samples (OS-Boot).

We first examine the internal inequality among winners. Figure 3 plots the change in positive LC from 2012 to 2014, *i.e.*, $\Delta L^\oplus = L^\oplus_{2014} - L^\oplus_{2012}$, and 95% CIs. We find that $L^\oplus_{2014}$ is closer to the

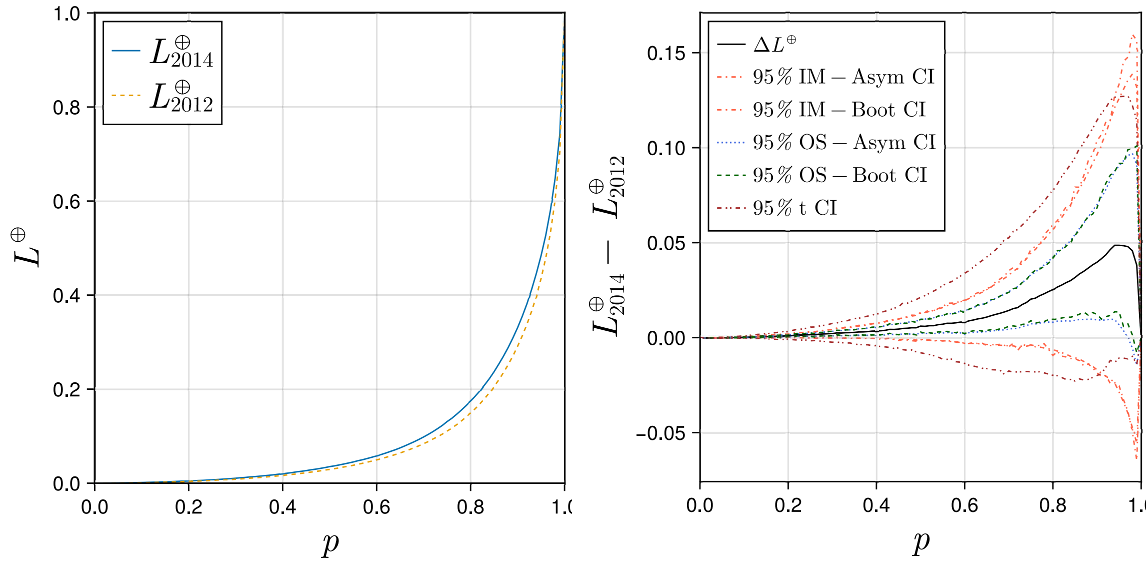

Figure 3. 2014 vs 2012: positively truncated Lorenz curves

diagonal line, which suggests a decrease in inequality level inside the group of winners. Asymptotic and bootstrap CIs based on overlapping samples (OS-Asym and OS-Boot) confirm this result, as zero falls outside the CIs. However, the intersection methods (IM-Asym and IM-Boot) give an opposite conclusion. Given the data structure, we believe that overlapping samples are a suitable assumption and thus prefer the findings from OS-Asym and OS-Boot, as IM can be conservative under such a scenario based on simulation results in the previous section. We observe similar results for $\Delta I^{\oplus}$ in Table 10. The estimate for $\Delta I^{\oplus}$ in Table 9 is only -0.0246, suggesting a relatively small decrease in inequality level.

We next study the internal inequality among the losers, as measured by $\Delta L^{\ominus}$ in Figure 4 and $\Delta I^{\ominus}$ in Table 10. We find that only the OS-Asym CI for $\Delta I^{\ominus}$ excludes zero, while all the other CIs contain zero. Given the simulation evidence in Section 5, the OS-Boot CI is generally more reliable than OS-Asym CI. Therefore, we cannot conclude that there is a significant change in internal inequality within the subgroup of losers.

Finally, we analyze the overall inequality based on ΔL^{s} in Figure 5 and ΔI^{p} in Table 10. The estimated signed LC displays a flat segment in the left panel of Figure 5, approximately from $p = 0.1$ to $p = 0.29$. At such points, the differentiability condition underlying Corollary 6.2 may fail. We therefore treat the pointwise intervals over the flat segment as descriptive unless the differentiability condition is verified or the affected points are excluded.

For $p < 0.1$, we note that asymptotic and bootstrap CIs with overlapping samples (OS-Asym and OS-Boot) exclude zero, indicating a change over time. We observe similar findings about ΔI^{p} by OS-Asym and OS-Boot CIs. In contrast, the CIs based on IMs (IM-Asym and IM-Boot) include zero, possibly due to the method's conservativeness. Note that, however, the magnitude of the change in inequality is economically small.

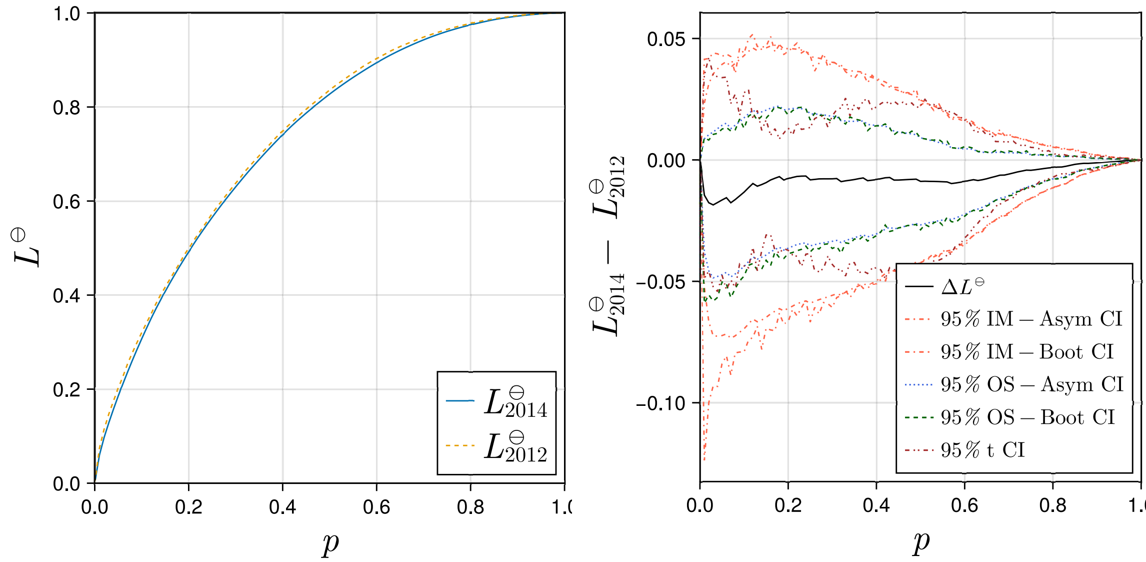

Figure 4. 2014 vs 2012: negatively truncated Lorenz curves

Table 9. 2014 vs 2012: estimates for extended Gini indices

I^*	I^*_{2012}	I^*_{2014}	ΔI^*
$I^{\oplus}$	0.8110	0.7864	-0.0246
$I^{\ominus}$	0.4836	0.4699	-0.0137
I^{p}	0.8899	0.8776	-0.0123

Note – $\Delta I^* = I^*_{2014} - I^*_{2012}$ denotes the extended Gini index change from 2012 to 2014, where I^* is one of the following indices: the positive truncated Gini index ($I^{\oplus}$), negative truncated Gini index ($I^{\ominus}$), and positive Gini index (I^p) in Definition 6.1.

The robust-t method yields wider CIs than IMs for several extended inequality index differences. Given the simulation results in Section 5, we view these robust-t CIs as less attractive in this application.

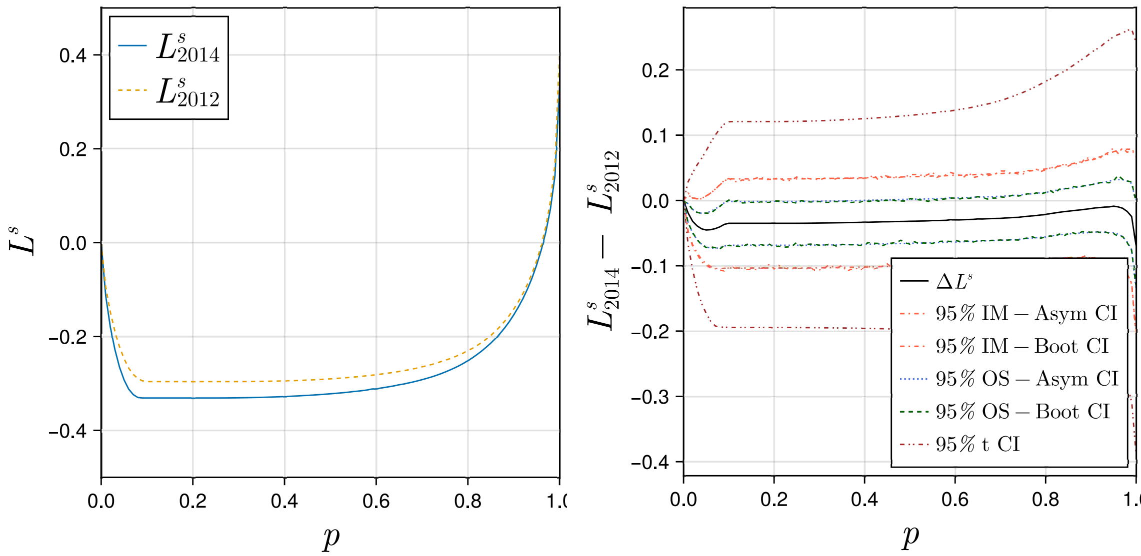

Figure 5. 2014 vs 2012: signed Lorenz curves

Table 10. 2014 vs 2012: 95% CIs for difference in extended Gini index

ΔI^*	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
$\Delta I^{\oplus}$	$[-0.0584,\ 0.0091]$	$[-0.0622,\ 0.0136]$	$[-0.0411,\ -0.0082]$	$[-0.0442,\ -0.0076]$	$[-0.0753,\ 0.0171]$
$\Delta I^{\ominus}$	$[-0.0685,\ 0.0410]$	$[-0.0721,\ 0.0412]$	$[-0.0231,\ -0.0044]$	$[-0.0445,\ 0.0166]$	$[-0.0441,\ 0.0137]$
ΔI^{p}	$[-0.0271,\ 0.0026]$	$[-0.0288,\ 0.0023]$	$[-0.0201,\ -0.0045]$	$[-0.0197,\ -0.0048]$	$[-0.0545,\ 0.0279]$

Note – $\Delta I^* = I^*_{2014} - I^*_{2012}$ denotes the extended Gini index change from 2012 to 2014, where I^* is one of the following indices: the positive truncated Gini index ($I^{\oplus}$), negative truncated Gini index ($I^{\ominus}$), and positive Gini index (I^p) in Definition 6.1. The IM-Boot and OS-Boot are based on 999 bootstrap repetitions each.

7. Extension to complex survey data

So far, we have assumed that researchers have a sample of micro-level data drawn independently and identically from a population distribution. However, many large-scale household surveys by design involve stratification and clustering. In this section, we briefly sketch the potential extensions of our methods to complex survey data.

In most surveys, the strata are chosen in advance and thus fixed. One can incorporate stratification using standard survey-sampling methods; see Cochran (1977) for a textbook-level treatment. Therefore, we mainly discuss how to extend our methods, especially the intersection approach, to clustered data. Our object remains the distribution of variables such as income. Therefore, concerns such as heavy tails will pertain to the distribution of the variables rather than, as analyzed in Chiang, Sasaki and Wang (2025), to the cluster sizes.

Cluster-robust inference has become increasingly popular over the past decade and is now employed routinely in many empirical microeconomics studies; see MacKinnon, Nielsen and Webb

(2023) for a recent survey. There are at least two ways to incorporate clusters into our framework.

First, both samples in Assumption 3.1 can be generalized to clustered data. In general, the intra-cluster correlation is positive, so when clusters are large, incorrectly assuming iid observations can lead to smaller asymptotic variances and thus unreliable inference (over-rejection of tests or insufficient coverage of CIs) [MacKinnon and MacKinnon (2016)].

Two distinct types of assumptions are available for the asymptotic theory of cluster-robust inference. The most common method is the "large number of clusters" approach, where the number of clusters, often denoted by G, tends to infinity [Djogbenou, MacKinnon and Nielsen (2019) and Hansen and Lee (2019)]. The other is the fixed-G or "small number of large clusters" method [Ibragimov and Müller (2010) and Ibragimov and Müller (2016)]. The bootstrap methods, especially the wild cluster restricted (WCR) bootstrap, often yield much more reliable inferences than asymptotic procedures [MacKinnon and Webb (2016) and MacKinnon (2023)]. The WCR bootstrap is usually based on the Rademacher distribution [Davidson and Flachaire (2008) and Djogbenou et al. (2019)]. Nevertheless, when the number of clusters is small, a six-point distribution is preferred [Webb (2023)]. We may further extend to a multi-stage sampling design using techniques in Zheng (2002) or Bhattacharya (2005).

Second, the clustered data can generate dependence between the two samples. For example, suppose that there are G groups, with group g containing a sample of n_g pairs of observations $(X_{1,gi}, X_{2,gi})$. One may conduct cluster-robust inference using either the "large number of clusters" or the "small number of large clusters" approach. However, the intersection methods (IMs) are also applicable. Furthermore, we can still employ IMs even when asymptotic inference can fail.

One example is cluster heterogeneity, which we can demonstrate by the simulation experiment under **DGP II-A**. We may view such a DGP for the "large number of clusters" approach with $G = n$. That is, each group contains only one pair of observations. Note that, due to sorting, the correlation within each pair varies dramatically across groups, except for $\rho = 0.95$, the highly positively correlated case. As reported in Table 7, the OS-based approach, which coincides trivially with the cluster-robust inference, fails to provide reliable CIs. However, the IM approach remains valid.

These issues, especially a full treatment of clustering, deserve further investigation and are beyond the scope of this paper.

8. Conclusion

Traditional inference methods for comparing welfare indices typically assume that samples are independent. However, this assumption can be problematic because income samples are often dependent due to the overlap between consecutive periods.

This paper considers a broad family of indices expressed as asymptotically linear Gaussian functionals and proposes inference methods for index differences with dependent samples. We first study generic dependent samples and provide asymptotic and bootstrap intersection methods that are robust to arbitrary sample dependence. We then study the overlapping samples, a specific type of dependent samples where the dependence arises solely from the matched pairs, and propose asymptotic and bootstrap confidence intervals for changes in indices. We additionally justify using influence functions to estimate the asymptotic variance and covariance consistently.

We evaluate the performance of the proposed methods through a series of simulation experiments. First, we find that sample dependence can substantially impact asymptotic variance. Therefore, our proposed methods, which utilize overlapping samples, can be more efficient than conventional procedures based on independent samples. Next, we analyze the effects of heavy tails and find that the asymptotic inference performs well for realistic distributions but poorly for distributions with heavy tails. The bootstrap method can alleviate this issue, except when the variance is substantial or nonexistent. Finally, the intersection methods prove to be reliable under generic dependent samples, particularly when the assumption of overlapping samples is inappropriate. The intersection methods are also reasonably efficient in certain instances, such as those involving extremely heavy-tailed distributions.

To illustrate our methods, we applied them to analyze the changes in financial inequality among Italian households from 2012 to 2014. The application also highlights the practical importance of using extended inequality measures in the presence of negative values.

We finally discuss potential extension to clustered data and multi-stage survey sampling. Moreover, we demonstrate that the intersection methods are also robust to some cases where cluster-robust inference fails.

References

Bahadur, R. R. and Savage, L. J. (1956), 'The nonexistence of certain statistical procedures in nonparametric problems', *The Annals of Mathematical Statistics* **27**(4), 1115–1122. Type: Journal Article.

Barrett, G. F. and Donald, S. G. (2009), 'Statistical inference with generalized Gini indices of inequality, poverty, and welfare', *Journal of Business & Economic Statistics* **27**(1), 1–17.

Beach, C. M. and Davidson, R. (1983), 'Distribution-free statistical inference with Lorenz curves and income shares', *Review of Economic Studies* **50**(4), 723–735.
*https://ideas.repec.org/a/oup/restud/v50y1983i4p723-735..html

Beran, R. (1988), 'Prepivoting test statistics: A bootstrap view of asymptotic refinements', *Journal of the American Statistical Association* **83**(403), 687–697.

Bhattacharya, D. (2005), 'Asymptotic inference from multi-stage samples', *Journal of Econometrics* **126**(1), 145–171.
*https://ideas.repec.org/a/eee/econom/v126y2005i1p145-171.html

Bhattacharya, D. (2007), 'Inference on inequality from household survey data', *Journal of Econometrics* **137**(2), 674–707.
*https://ideas.repec.org/a/eee/econom/v137y2007i2p674-707.html

Biewen, M. (2002), 'Bootstrap inference for inequality, mobility and poverty measurement', *Journal of Econometrics* **108**(2), 317–342.
*https://www.sciencedirect.com/science/article/pii/S0304407601001385

Bouezmarni, T. and Dufour, J.-M. (2024), Winners and losers: Extended Lorenz curves and Gini coefficients for possibly negative variables, Working paper, McGill University. Working paper.

Chakravarty, S. R. (1988), 'Extended Gini indices of inequality', *International Economic Review* **29**(1), 147.

Chiang, H. D., Sasaki, Y. and Wang, Y. (2025), Genuinely robust inference for clustered data, Working paper, arXiv. Working paper.

Cochran, W. G. (1977), *Sampling Techniques*, A Wiley publication in applied statistics, 3. ed. edn, Wiley, New York. Bibliography p. 400 - 411.

Cowell, F. A. and Flachaire, E. (2015), Statistical methods for distributional analysis, *in* A. B. Atkinson and F. Bourguignon, eds, 'Handbook of Income Distribution', Vol. 2 of *Handbook of Income Distribution*, Elsevier, chapter 6, pp. 359–465.
*https://www.sciencedirect.com/science/article/pii/B9780444594280000072

Cowell, F. A. and Victoria-Feser, M.-P. (1996), 'Robustness properties of inequality measures', *Econometrica* **64**(1), 77–101.
*http://www.jstor.org/stable/2171925

Davidson, R. (2009), 'Reliable inference for the Gini index', *Journal of Econometrics* **150**(1), 30–40.
*https://linkinghub.elsevier.com/retrieve/pii/S0304407609000323

Davidson, R. (2010), 'Innis lecture: Inference on income distributions', *Canadian Journal of Economics/Revue canadienne d'économique* **43**(4), 1122–1148.

Davidson, R. (2012), 'Statistical inference in the presence of heavy tails', *The Econometrics Journal* **15**(1), C31–C53.

Davidson, R. and Flachaire, E. (2007), 'Asymptotic and bootstrap inference for inequality and poverty measures', *Journal of Econometrics* **141**(1), 141–166. Semiparametric methods in econometrics.
*https://www.sciencedirect.com/science/article/pii/S0304407607000115

Davidson, R. and Flachaire, E. (2008), 'The wild bootstrap, tamed at last', *Journal of Econometrics* **146**(1), 162–169.

Davidson, R. and MacKinnon, J. G. (2000), 'Bootstrap tests: How many bootstraps?', *Econometric Reviews* **19**(1), 55–68.

Davidson, R. and MacKinnon, J. G. (2004), *Econometric Theory and Methods*, Oxford University Press, New York. Literaturverz. S. 702-721.

Djogbenou, A. A., MacKinnon, J. G. and Nielsen, M. Ø. (2019), 'Asymptotic theory and wild bootstrap inference with clustered errors', *Journal of Econometrics* **212**(2), 393–412.

Donaldson, D. and Weymark, J. A. (1980), 'A single-parameter generalization of the Gini indices of inequality', *Journal of Economic Theory* **22**(1), 67–86.

Dufour, J.-M., Flachaire, E. and Khalaf, L. (2019), 'Permutation tests for comparing inequality measures', *Journal of Business & Economic Statistics* **37**(3), 457–470.
*https://doi.org/10.1080/07350015.2017.1371027

Dufour, J.-M., Flachaire, E., Khalaf, L. and Zalghout, A. (2024), 'Identification-robust methods for comparing inequality with an application to regional disparities', *The Journal of Economic Inequality* **22**(2), 433–452.
*https://ideas.repec.org/p/cir/cirwor/2020s-23.html

Dufour, J.-M. and He, T. (2024*a*), Finite-sample distribution-free confidence intervals for comparing means of bounded variables with application to poverty measures, Working paper, McGill University.

Dufour, J.-M. and He, T. (2024*b*), Measuring inequality for winners and losers: Extended Lorenz curves and Gini indices for possibly negative variables, Working paper, McGill University. Working paper.

Dufour, J.-M. and Torrès, O. (1998), Union-intersection and sample-split methods in econometrics with applications to MA and SURE models, *in* A. Ullah and D. E. A. Giles, eds, 'Handbook of Applied Economic Statistics', CRC Press, chapter 14, pp. 485–491.

Gastwirth, J. L. (1971), 'A general definition of the Lorenz curve', *Econometrica* **39**(6), 1037.
*https://www.jstor.org/stable/1909675?origin=crossref

Hall, P. (1992), *The Bootstrap and Edgeworth Expansion*, Springer Series in Statistics, Springer, New York.

Hampel, F. R. (1974), 'The influence curve and its role in robust estimation', *Journal of the American Statistical Association* **69**(346), 383–393.

Hansen, B. E. and Lee, S. (2019), 'Asymptotic theory for clustered samples', *Journal of Econometrics* **210**(2), 268–290.
*https://linkinghub.elsevier.com/retrieve/pii/S0304407619300247

Ibragimov, R., Kattuman, P. and Skrobotov, A. (2025), 'Robust inference on income inequality: T-statistic based approach', *Econometric Reviews* **44**(4), 384–415.

Ibragimov, R. and Müller, U. K. (2010), 't-statistic based correlation and heterogeneity robust inference', *Journal of Business & Economic Statistics* **28**(4), 453–468.

Ibragimov, R. and Müller, U. K. (2016), 'Inference with few heterogeneous clusters', *The Review of Economics and Statistics* **98**(1), 83–96.

Kakwani, N. and Lambert, P. J. (1998), 'On measuring inequity in taxation: A new approach', *European Journal of Political Economy* **14**(2), 369–380.
*https://linkinghub.elsevier.com/retrieve/pii/S0176268098000123

Kleiber, C. and Kotz, S. (2003), *Statistical Size Distributions in Economics and Actuarial Sciences*, Vol. 470, Wiley, New York.

Lehmann, E. L. and Romano, J. P. (2022), *Testing Statistical Hypotheses*, Springer texts in statistics, 3. ed., correct. at 4. print. edn, Springer, New York.

MacKinnon, J. G. (2023), 'Fast cluster bootstrap methods for linear regression models', *Econometrics and Statistics* **26**, 52–71.

MacKinnon, J. G. and MacKinnon, J. G. (2016), 'Inference with large clustered datasets', *Queen's Economics Department Working Paper* (2110-2018-4501), 18.
*http://ageconsearch.umn.edu/record/274691

MacKinnon, J. G., Nielsen, M. Ø. and Webb, M. D. (2023), 'Cluster-robust inference: A guide to empirical practice', *Journal of Econometrics* **232**(2), 272–299.

MacKinnon, J. G. and Webb, M. D. (2016), 'Wild bootstrap inference for wildly different cluster sizes', *Journal of Applied Econometrics* **32**(2), 233–254.

McDonald, J. B. (1984), 'Some generalized functions for the size distribution of income', *Econometrica* **52**(3), 647.

Mills, J. A. and Zandvakili, S. (1997), 'Statistical inference via bootstrapping for measures of inequality', *Journal of Applied Econometrics* **12**(2), 133–150.

Raffinetti, E., Siletti, E. and Vernizzi, A. (2015), 'On the Gini coefficient normalization when attributes with negative values are considered', *Statistical Methods & Applications* **24**(3), 507–521.

Raffinetti, E., Siletti, E. and Vernizzi, A. (2017), 'Analyzing the effects of negative and non-negative values on income inequality: Evidence from the Survey of Household Income and Wealth of the Bank of Italy (2012)', *Social Indicators Research* **133**(1), 185–207.

Scarpa, S., Ferrante, M. R. and Sperlich, S. (2025), 'Inference for the quantile ratio inequality index in the context of survey data', *Journal of Survey Statistics and Methodology* pp. 1–21.

Shao, J. and Tu, D. (1995), *The Jackknife and Bootstrap*, Springer, New York.

Shi, X. (1986), 'A note on bootstrapping the U-statistics', *Chinese Journal of Applied Probability and Statistics* **2**(2), 144–148.

Wasserman, L. (2006), *All of Nonparametric Statistics*, Springer Texts in Statistics, Springer, New York.

Webb, M. D. (2023), 'Reworking wild bootstrap-based inference for clustered errors', *Canadian Journal of Economics/Revue canadienne d'économique* **56**(3), 839–858.

Yitzhaki, S. and Schechtman, E. (2013), *The Gini Methodology*, Vol. 272 of *Springer Series in Statistics*, Springer, New York.

Zhang, S., Martin, R. D. and Christidis, A. A. (2021), 'Influence functions for risk and performance estimators', *Journal of Mathematical Finance* **11**(01), 15–47.

Zheng, B. (2002), 'Testing Lorenz curves with non-simple random samples', *Econometrica* **70**(3), 1235–1243.

Zheng, B. (2004), 'Poverty comparisons with dependent samples', *Journal of Applied Econometrics* **19**(3), 419–428.
*https://onlinelibrary.wiley.com/doi/abs/10.1002/jae.779

Zheng, B. and J. Cushing, B. (2001), 'Statistical inference for testing inequality indices with dependent samples', *Journal of Econometrics* **101**(2), 315–335.
*https://linkinghub.elsevier.com/retrieve/pii/S0304407600000877

Nonparametric methods for comparing distribution functionals for dependent samples with applications to welfare indices

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August 30, 2026

Technical appendix

This appendix includes proofs of propositions and additional simulation results.

A. Proofs

A.1. Intersection methods in Section 3

Dufour and He (2024*a*) have proved the results for finite samples. We make slight modifications to adapt to the asymptotic scenario. The following proof works for both Proposition 3.1 and 3.2.

PROOF OF PROPOSITION 3.1. Take the asymptotic CI as an example, as the bootstrap CI can be similarly derived. Under Assumption 3.1, we have the asymptotic CI for $\Delta\theta$ in (4.41). Then, we have

$$\Pr\left(L_{n_1} - U_{n_2} \leq \Delta\theta \leq U_{n_1} - L_{n_2}\right) \geq \Pr\left(\{L_{n_1} \leq \theta_1 \leq U_{n_1}\} \bigcap \{L_{n_2} \leq \theta_2 \leq U_{n_2}\}\right) \tag{A.1}$$

$$\geq 1 - \sum_{k=1}^{2} \Pr\left(\theta_k \notin [L_{n_k}, U_{n_k}]\right) \geq 1 - (\alpha_1 + \alpha_2) \geq 1 - \alpha \tag{A.2}$$

where the second inequality is due to the Bonferroni inequality. When two samples are independent, the Bonferroni inequality can be tight. As the limiting confidence interval of $[L_{n_k}, U_{n_k}]$ is $(1-\alpha_k)$, we have the following result:

$$\begin{aligned} &\Pr\left(\{L_1 \leq \theta_1 \leq U_1\} \bigcap \{L_2 \leq \theta_2 \leq U_2\}\right) \\ =&\Pr\left(L_1 \leq \theta_1 \leq U_1\right) \Pr\left(L_2 \leq \theta_2 \leq U_2\right) = (1-\alpha_1)(1-\alpha_2) \\ =&1 - (\alpha_1 + \alpha_2) + \alpha_1\alpha_2 = 1 - \alpha. \end{aligned} \tag{A.3}$$

□

A.2. Asymptotic distribution of index difference in Section 4.1

PROOF OF PROPOSITION 4.1. To simplify notations, let $\psi(X_{ki}) = \psi(X_{ki};\theta,F_k)$, $\sigma_k^2 = \mathrm{Var}[\psi(X_{ki})]$ and $\sigma_{12} = \sigma_{21} = \mathrm{Cov}[\psi(X_{1i}),\psi(X_{2i})]$. Under the Assumption 4.1, we have

$$\begin{aligned}\sqrt{N}(\hat{\theta}_1-\theta_1) &= \sqrt{\frac{n_2}{n_1+n_2}}\frac{1}{\sqrt{n_1}}\sum_{i=1}^{n_1}\psi(X_{1i})+o_p(1) \xrightarrow{d} \mathrm{N}\left(0,\,(1-\eta_1)\sigma_1^2\right)\\ \sqrt{N}(\hat{\theta}_2-\theta_2) &= \sqrt{\frac{n_1}{n_1+n_2}}\frac{1}{\sqrt{n_2}}\sum_{i=1}^{n_2}\psi(X_{2i})+o_p(1) \xrightarrow{d} \mathrm{N}\left(0,\,\eta_1\sigma_2^2\right)\end{aligned} \tag{A.4}$$

We assume $m \neq 0$ as the derivation for $m = 0$ is straightforward. Under the additional Assumption 4.2, we have:

$$\begin{aligned}\sqrt{N}(\hat{\theta}_1-\theta_1) &= \sqrt{\frac{n_2}{n_1+n_2}}\sqrt{\frac{m}{n_1}}\frac{1}{\sqrt{m}}\left[\sum_{i=1}^{m}\psi(X_{1i})+\sum_{i=m+1}^{n_1}\psi(X_{1i})\right]+o_p(1)\,,\\ \sqrt{N}(\hat{\theta}_2-\theta_2) &= \sqrt{\frac{n_1}{n_1+n_2}}\sqrt{\frac{m}{n_2}}\frac{1}{\sqrt{m}}\left[\sum_{i=1}^{m}\psi(X_{2i})+\sum_{i=m+1}^{n_2}\psi(X_{2i})\right]+o_p(1)\,.\end{aligned} \tag{A.5}$$

By Assumption 4.1, only matched pairs contribute to the covariance term. So we have asymptotic covariance $\sqrt{(1-\eta_1)\eta_1\lambda_1\lambda_2}\sigma_{12}$. Therefore,

$$\begin{aligned}&\sqrt{N}\left(\begin{bmatrix}\hat{\theta}_1\\ \hat{\theta}_2\end{bmatrix}-\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\right) = \begin{bmatrix}\sqrt{\frac{n_2}{n_1+n_2}}\sqrt{\frac{m}{n_1}}\frac{1}{\sqrt{m}}\sum_{i=1}^{n_1}\psi(X_{1i})\\ \sqrt{\frac{n_1}{n_1+n_2}}\sqrt{\frac{m}{n_2}}\frac{1}{\sqrt{m}}\sum_{i=1}^{n_2}\psi(X_{2i})\end{bmatrix}+\begin{bmatrix}o_p(1)\\ o_p(1)\end{bmatrix}\\ &\xrightarrow{d} \mathrm{N}\left(\begin{bmatrix}0\\ 0\end{bmatrix},\begin{bmatrix}(1-\eta_1)\sigma_1^2 & \sqrt{(1-\eta_1)\eta_1\lambda_1\lambda_2}\sigma_{12}\\ \sqrt{(1-\eta_1)\eta_1\lambda_1\lambda_2}\sigma_{21} & \eta_1\sigma_2^2\end{bmatrix}\right).\end{aligned} \tag{A.6}$$

The desired result then follows by continuous mapping theorem. □

A.3. Asymptotic variance estimation in Section 4.2

The consistency proof often relies on specific conditions for given indices. Nevertheless, we can gain insight into how to prove consistency based on techniques presented in Barrett and Donald (2009), particularly Lemma 5 in their appendix. We can also extend those results to an estimator for the covariance between $\hat{\psi}_1$ and $\hat{\psi}_2$.

Lemma A.1 CONSISTENT ESTIMATOR FOR ASYMPTOTIC VARIANCE. *Let $\hat{\psi}_{ki}$, $k = 1,2$ be an estimate for ψ_{ki} evaluated at X_{ki} with ψ_k in* (2.1) *such that*

$$|\hat{\psi}_{ki}-\psi_{ki}| \leq \alpha_k\hat{A}_k+\rho_{ki}\hat{B}_k+\beta_k\hat{P}_{ki}+\omega_{ki}\hat{W}_{ki}. \tag{A.7}$$

Assume that the following conditions hold:

(1) $|\alpha_k| = O_p(1)$ *and* $\hat{A}_k = o_p(1)$,

(2) $\left|\frac{1}{n_k}\sum_{i=1}^{n_k}\rho_{ki}^2\right| = O_p(1)$ *and* $\hat{B}_k = o_p(1)$,

(3) $|\beta_k| = O_p(1)$ *and* $\frac{1}{n_k}\sum_{i=1}^{n_k}\hat{P}_{ki}^2 = o_p(1)$,

(4) $|\frac{1}{n_k}\sum_{i=1}^{n_k}\omega_{ki}^4| = O_p(1)$ *and* $\frac{1}{n_k}\sum_{i=1}^{n_k}\hat{W}_{ki}^2 = o_p(1)$,

(5) $\frac{1}{n_k}\sum_{i=1}^{n_k}\psi_{ki}^2 = O_p(1)$.

(1) *[Barrett and Donald (2009, Lemma 5)] We then have*

$$\frac{1}{n_k}\sum_{i=1}^{n_k}\hat{\psi}_{ki}^2 - \frac{1}{n_k}\sum_{i=1}^{n_k}\psi_{ki}^2 = o_p(1). \tag{A.8}$$

(2) *For matched pairs* $\{(X_{1i}, X_{2i})\}_{i=1}^m$, *we have*

$$\frac{1}{m}\sum_{i=1}^{m}\hat{\psi}_{1i}\hat{\psi}_{2i} - \frac{1}{m}\sum_{i=1}^{m}\psi_{1i}\psi_{2i} = o_p(1). \tag{A.9}$$

PROOF OF LEMMA A.1 Barrett and Donald (2009, Lemma 5) establish (A.8). We include this result here for completeness. To prove (A.9), note that

$$\begin{aligned}
\left|\frac{1}{m}\sum_{i=1}^{m}\hat{\psi}_{1i}\hat{\psi}_{2i} - \frac{1}{m}\sum_{i=1}^{m}\psi_{1i}\psi_{2i}\right| =& \left|\frac{1}{m}\sum_{i=1}^{m}(\hat{\psi}_{1i}-\psi_{1i})(\hat{\psi}_{2i}-\psi_{2i})\right. \\
&\left. + \frac{1}{m}\sum_{i=1}^{m}(\hat{\psi}_{1i}-\psi_{1i})\psi_{2i} + \frac{1}{m}\sum_{i=1}^{m}\psi_{1i}(\hat{\psi}_{2i}-\psi_{2i})\right| \\
\leq& \left[\frac{1}{m}\sum_{i=1}^{m}(\hat{\psi}_{1i}-\psi_{1i})^2\right]^{1/2}\left[\frac{1}{m}\sum_{i=1}^{m}(\hat{\psi}_{2i}-\psi_{2i})^2\right]^{1/2} \\
&+ \left[\frac{1}{m}\sum_{i=1}^{m}(\hat{\psi}_{1i}-\psi_{1i})^2\right]^{1/2}\left(\frac{1}{m}\sum_{i=1}^{m}\psi_{2i}^2\right)^{1/2} \\
&+ \left(\frac{1}{m}\sum_{i=1}^{m}\psi_{1i}^2\right)^{1/2}\left[\frac{1}{m}\sum_{i=1}^{m}(\hat{\psi}_{2i}-\psi_{2i})^2\right]^{1/2} \\
=& o_p(1),
\end{aligned} \tag{A.10}$$

where the inequality is due to the Cauchy-Schwarz inequality and the last equality is implied by conditions (i) - (v). □

A.4. Bootstrap inference in Section 4.3

PROOF OF LEMMA 4.2 By Assumption 4.1 and the bootstrap algorithm, one can establish the following asymptotic linear representation conditional on the original data $\{X_{1i}\}_{i=1}^{n_1}$ and $\{X_{2i}\}_{i=1}^{n_2}$; see, for example, Shao and Tu (1995).

$$\sqrt{n_k}(\hat{\theta}_k^* - \hat{\theta}_k) = \frac{1}{\sqrt{n_k}} \sum_{i=1}^{n_k} \psi(X_{ki}^*; \theta, \hat{F}_k) + o_p(1) \tag{A.11}$$

where $\hat{\theta}_k^*$ denotes the estimate based on a bootstrap sample $\{X_{ki}^*\}_{i=1}^{n_k}$ and $\hat{F}_k$ is the EDF for F_k. Conditional on the original data, we have, under additional Assumption 4.2,

$$\mathbb{E}\left[\mathrm{IF}(X_{ki}^*; \theta, \hat{F}_k) \mid P_n\right] = \frac{1}{n_k} \sum_{i=1}^{n_k} \psi(X_{ki}; \theta, \hat{F}_k) \xrightarrow{a.s.} \mathbb{E}\left[\psi(X_{ki}; \theta, F_k)\right] = 0\,, \tag{A.12}$$

$$\mathbb{E}\left[\psi^2(X_{ki}^*; \theta, \hat{F}_k) \mid P_n\right] = \frac{1}{n_k} \sum_{i=1}^{n_k} \psi^2(X_{ki}; \theta, \hat{F}_k) \xrightarrow{a.s.} \mathbb{E}\left[\psi^2(X_{ki}; \theta, F_k)\right] < \infty\,. \tag{A.13}$$

Then by the central limit theorem, we have

$$\frac{1}{\sqrt{n_k}} \sum_{i=1}^{n_k} \psi(X_{ki}^*; \theta, \hat{F}_k) \xrightarrow{d} \mathrm{N}(0, \Omega_k), \quad k = 1, 2, \tag{A.14}$$

$$\frac{1}{\sqrt{m}} \sum_{i=1}^{m} \begin{bmatrix} \psi(X_{1i}^*; \theta, \hat{F}_1) \\ \psi(X_{2i}^*; \theta, \hat{F}_2) \end{bmatrix} \xrightarrow{d} \mathrm{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} \right), \tag{A.15}$$

where $\sigma_k^2 = \mathrm{Var}[\psi(X_{ki}; \theta, F_k)]$ and $\sigma_{12} = \sigma_{21} = \mathrm{Cov}[\psi(X_{1i}; \theta, F_1), \psi(X_{2i}; \theta, F_2)]$. Similar to the proof of Proposition 4.1, we have

$$\sqrt{N}\left(\begin{bmatrix} \hat{\theta}_1^* \\ \hat{\theta}_2^* \end{bmatrix} - \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} \right) \xrightarrow{d} \mathrm{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} (1-\eta_1)\sigma_1^2 & \sqrt{(1-\eta_1)\eta_1\lambda_1\lambda_2}\sigma_{12} \\ \sqrt{(1-\eta_1)\eta_1\lambda_1\lambda_2}\sigma_{21} & \eta_1\sigma_2^2 \end{bmatrix} \right)\,. \tag{A.16}$$

By continuous mapping theorem, we show the validity of the bootstrap method. □

A.5. Proof in Section 6

PROOF OF COROLLARY 6.2. Dufour and He (2024*b*) shows that the estimates for extended LCs and Gini indices satisfy Definition 2.1 with the influence function given in the main text. Under the overlapping samples (*i.e.*, Assumption 4.1), we can show that the Assumption 4.2 is satisfied by the standard multivariate central limit theorem. We then can apply Proposition 4.1 to conclude the results. □

B. Additional simulation results

B.1. Additional simulation results in Section 5.1

We present simulated performance (coverage and width) of intersection methods, overlapping-sample methods and (incorrect) independent-sample methods under additional values of $\rho \in \{\mp 0.5, 0, 0.99\}$.

Table 11 - Performance of proposed and conventional 95% confidence intervals for mean difference under additional values of ρ

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
		Intersection method				Overlapping samples				Independent samples (incorrect)			
ρ	λ	Coverage (%)		Width		Coverage (%)		Width		Coverage (%)		Width	
		Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot
-0.5	0.1	99.7	99.7	0.0534	0.0536	94.7	94.2	0.0338	0.0339	94.2	94.4	0.0331	0.0333
	0.5	99.7	99.6	0.0534	0.0536	94.3	94.3	0.0364	0.0366	91.9	91.6	0.0331	0.0332
	0.9	99.6	99.7	0.0535	0.0537	95.6	96	0.0389	0.039	90.6	90.4	0.0331	0.0333
0	0.1	100	100	0.0535	0.0537	95.3	95.1	0.0331	0.0333	95.3	94.9	0.0331	0.0333
	0.5	100	99.9	0.0534	0.0536	95.5	94.7	0.0331	0.0333	95.5	95	0.0331	0.0333
	0.9	99.6	99.6	0.0534	0.0536	95	94.8	0.0331	0.0332	95.4	94.6	0.0331	0.0333
0.5	0.1	100	100	0.0534	0.0536	95.8	95.3	0.0323	0.0324	96.2	96	0.0331	0.0332
	0.5	100	99.9	0.0534	0.0537	95.3	95.7	0.029	0.0292	97.6	97.8	0.0331	0.0333
	0.9	100	100	0.0534	0.0536	96	95.7	0.0252	0.0254	99.1	98.8	0.0331	0.0333
0.99	0.1	99.9	99.8	0.0535	0.0537	94.4	94.4	0.0315	0.0317	95.2	95.2	0.0332	0.0333
	0.5	100	100	0.0534	0.0536	94.7	94.3	0.0237	0.0238	99	99.2	0.0331	0.0333
	0.9	100	100	0.0534	0.0537	95.4	95.8	0.0117	0.0118	100	100	0.0331	0.0332

Note – The table provides coverage rates and widths of 95% confidence intervals for the mean differences. The data are generated according to DGP: $F_1 = \text{SM}(1, 1.6971, 8.3679)$, $F_2 = \text{SM}(0.4, 2.8, 1.7)$, $\rho \in \{-0.5, 0, 0.5, 0.99\}$ in (5.3), overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$. The sample sizes are both 1000. Columns (1) - (4) are performance by asymptotic intersection method (Asym), bootstrap intersection method (Boot) in Section 3. Columns (5) - (8) are performance by asymptotic (Asym) and bootstrap (Boot) methods with overlapping samples in Section 4, using $\hat{\sigma}^D_\Delta$ in (5.5). Columns (9) - (12) are performance by asymptotic (Asym) and bootstrap (Boot) methods if incorrectly ignoring the sample dependence, using $\hat{\sigma}^I_\Delta$ in (5.6). The results are based on 1000 replications with 999 bootstrap repetitions each. The bold font indicates cases where coverage rates largely deviate from 95%.

Table 12. Performance of proposed and conventional 95% confidence intervals for Gini index difference under additional values of ρ

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
		Intersection method				Overlapping samples				Independent samples (incorrect)			
ρ	λ	Coverage (%)		Width		Coverage (%)		Width		Coverage (%)		Width	
		Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot
-0.5	0.1	99.8	99.9	0.0665	0.0675	94.8	94.5	0.0409	0.0412	95	94.8	0.0412	0.0417
	0.5	99.6	99.6	0.0666	0.0677	94.6	93.7	0.0396	0.04	95.6	95.7	0.0413	0.0417
	0.9	100	99.9	0.0667	0.0678	95.5	95.3	0.0382	0.0385	97.3	97.2	0.0413	0.0417
0	0.1	99.9	99.8	0.0666	0.0676	95.8	95.9	0.0413	0.0417	95.8	95.7	0.0413	0.0417
	0.5	99.5	99.5	0.0667	0.0678	94.9	94.2	0.0413	0.0417	95.1	94.6	0.0413	0.0418
	0.9	99.8	99.9	0.0666	0.0677	94.8	95	0.0413	0.0417	94.7	94.6	0.0413	0.0417
0.5	0.1	99.9	99.9	0.0666	0.0675	94.8	94.3	0.0408	0.0412	95.3	94.5	0.0412	0.0417
	0.5	100	99.9	0.0666	0.0678	96.8	96.4	0.0389	0.0394	97.8	97.4	0.0413	0.0418
	0.9	99.9	99.9	0.0666	0.0677	94.1	94.3	0.0369	0.0373	97.1	97.1	0.0413	0.0417
0.99	0.1	99.9	99.9	0.0667	0.0677	94.5	94.3	0.0393	0.0397	96	95.9	0.0413	0.0416
	0.5	100	100	0.0665	0.0674	95.8	95.5	0.0302	0.0306	99.2	99.1	0.0412	0.0416
	0.9	100	100	0.0665	0.0675	95	94.8	0.017	0.0171	100	100	0.0412	0.0416

Note – The table provides coverage rates and widths of 95% confidence intervals for the Gini index differences. The data are generated according to DGP: $F_1 = \mathrm{SM}(1, 1.6971, 8.3679)$, $F_2 = \mathrm{SM}(0.4, 2.8, 1.7)$, $\rho \in \{-0.5, 0, 0.5, 0.99\}$ in (5.3), overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$. The sample sizes are both 1000. Columns (1) - (4) are performance by asymptotic intersection method (Asym), bootstrap intersection method (Boot) in Section 3. Columns (5) - (8) are performance by asymptotic (Asym) and bootstrap (Boot) methods with overlapping samples in Section 4, using $\hat{\sigma}^D_\Delta$ in (5.5). Columns (9) - (12) are performance by asymptotic (Asym) and bootstrap (Boot) methods if incorrectly ignoring the sample dependence, using $\hat{\sigma}^I_\Delta$ in (5.6). The results are based on 1000 replications with 999 bootstrap repetitions each. The bold font indicates cases where coverage rates largely deviate from 95%.

Table 13. Performance of proposed and conventional 95% confidence intervals
for Lorenz curve difference at percentile 0.5 under additional values of ρ

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
		Intersection method				Overlapping samples				Independent samples (incorrect)			
ρ	λ	Coverage (%)		Width		Coverage (%)		Width		Coverage (%)		Width	
		Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot	Asym	Boot
-0.5	0.1	99.9	99.9	0.0451	0.0459	94.4	94.5	0.0277	0.0282	94.5	94.8	0.0279	0.0284
	0.5	99.7	99.7	0.0451	0.0459	94.4	94.8	0.0267	0.0272	95.5	95.4	0.0279	0.0284
	0.9	100	100	0.0451	0.046	96.3	96.6	0.0257	0.0262	97.6	97.8	0.0279	0.0284
0	0.1	99.9	99.9	0.0451	0.0459	95.7	96	0.0279	0.0284	95.8	96	0.0279	0.0285
	0.5	99.8	99.8	0.0451	0.0459	94.2	94.7	0.0279	0.0284	94.2	95.2	0.0279	0.0284
	0.9	99.9	99.9	0.0451	0.0459	95.2	95.3	0.0279	0.0284	95.5	95.9	0.0279	0.0283
0.5	0.1	99.9	99.9	0.0451	0.0459	94.4	94.4	0.0276	0.0281	94.4	94.4	0.0279	0.0285
	0.5	100	100	0.0451	0.046	96.4	96.3	0.0264	0.0269	97.4	97.3	0.0279	0.0284
	0.9	100	99.9	0.0451	0.0459	95	95.2	0.025	0.0255	97.2	97.3	0.0279	0.0284
0.99	0.1	99.8	100	0.0451	0.0459	94.8	94.8	0.0266	0.0271	96	96.1	0.0279	0.0284
	0.5	100	100	0.045	0.0458	95.1	95.5	0.0204	0.021	99.4	99.3	0.0279	0.0284
	0.9	100	100	0.0451	0.0459	94.7	96.3	0.0114	0.0122	100	100	0.0279	0.0284

Note – The table provides coverage rates and widths of 95% confidence intervals for the Lorenz curve difference at percentile 0.5. The data are generated according to DGP: $F_1 = \mathrm{SM}(1, 1.6971, 8.3679)$, $F_2 = \mathrm{SM}(0.4, 2.8, 1.7)$, $\rho \in \{-0.5, 0, 0.5, 0.99\}$ in (5.3), overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$. The sample sizes are both 1000. Columns (1) - (4) are performance by asymptotic intersection method (Asym), bootstrap intersection method (Boot) in Section 3. Columns (5) - (8) are performance by asymptotic (Asym) and bootstrap (Boot) methods with overlapping samples in Section 4, using $\hat{\sigma}^D_\Delta$ in (5.5). Columns (9) - (12) are performance by asymptotic (Asym) and bootstrap (Boot) methods if incorrectly ignoring the sample dependence, using $\hat{\sigma}^I_\Delta$ in (5.6). The results are based on 1000 replications with 999 bootstrap repetitions each. The bold font indicates cases where coverage rates largely deviate from 95%.

B.2. Simulation results of robust-t method for alternative nominal levels

We examine performance of robust-t method under alternative nominal levels of 90% and 85% under DGP I-A where the tails are not heavy. We find that the coverage is clearly below the associated nominal level.

Table 14. DGP I-A: Coverage and width of confidence intervals
for the mean difference based on robust-t method at 90% and 85% levels

Nominal level(%)	λ	n	Coverage IM-Asym	Coverage OS-Asym	Coverage t	Width IM-Asym	Width OS-Asym	Width t
90	0.1	100	99	89.8	93.7	0.146	0.0909	0.12
		200	99.1	89	93.5	0.1039	0.0645	0.0849
		500	99.3	90.5	93.6	0.066	0.0409	0.054
		1000	99.2	89.8	93.5	0.0467	0.0289	0.0382
	0.5	100	98	89.8	89.7	0.1464	0.1037	0.1191
		200	98	89.8	90	0.1039	0.0733	0.0843
		500	97.9	90.3	90.4	0.0659	0.0464	0.0531
		1000	98.1	90.4	90.4	0.0467	0.0328	0.0377
	0.9	100	95.7	89.3	85.3	0.1463	0.1145	0.1176
		200	96.2	89.8	85.2	0.104	0.0812	0.0833
		500	96.1	89.8	85.4	0.066	0.0514	0.0528
		1000	96.5	90.3	86	0.0467	0.0364	0.0374
85	0.1	100	98.4	84.9	88.9	0.1327	0.0796	0.0985
		200	98.2	85.5	88.4	0.0944	0.0564	0.0694
		500	98.2	84.8	88.7	0.06	0.0358	0.0442
		1000	98.4	84.9	88.7	0.0424	0.0253	0.0312
	0.5	100	96.1	84.7	84	0.1328	0.0906	0.0972
		200	96.8	84.7	84	0.0944	0.0642	0.0693
		500	96.5	85	83.8	0.0599	0.0406	0.0436
		1000	96.5	84.7	83.1	0.0424	0.0287	0.0308
	0.9	100	93.9	84.3	78.2	0.1329	0.1002	0.0967
		200	94.4	84.4	78.8	0.0944	0.071	0.0681
		500	94	84.5	78.1	0.0599	0.0449	0.0432
		1000	94.5	84.6	78.7	0.0424	0.0318	0.0305

Note – The table provides coverage rates and widths of confidence intervals of level 90% and 85% for mean difference by asymptotic intersection method (IM-Asym), asymptotic method with overlapping samples (OS-Asym) and robust-t method (t) with $q = 4$ groups for both samples. The data are generated according to DGP I-A for given sample size n, overlap portion λ. The numbers are based on 10000 replications.

B.3. Additional simulation results on effects of heavy tails

We provide additional simulation results under DGP I-A, DGP I-B and DGP I-C with alternative values of ρ in Gaussian copula 5.3. In particular, we here consider the following DGPs.

DGP I-a: $F_1 = \text{SM}(1, 1.6971, 8.3679)$, $F_2 = \text{SM}(0.4, 2.8, 1.7)$, $\rho \in \{\mp 0.5, 0, 0.99\}$ in (5.3), stability index of F_2 is $\beta_2 = 4.76$, overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$.

DGP I-b: $F_1 = \text{SM}(1, 1.6971, 8.3679)$, $F_2 = \text{SM}(0.4, 2, 1.5)$, $\rho \in \{\mp 0.5, 0, 0.99\}$ in (5.3), stability index of F_2 is $\beta_2 = 3$, overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$.

DGP I-c: $F_1 = \text{SM}(1, 1.6971, 8.3679)$, $F_2 = \text{SM}(0.4, 1.4, 1.5)$, $\rho \in \{\mp 0.5, 0, 0.99\}$ in (5.3), stability index of F_2 is $\beta_2 = 2.1$, overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$.

This section provides a numerical exercise on the performance of the proposed methods when the tail is extremely heavy, such that the variance does not even exist. Under such a scenario, it is challenging to obtain valid asymptotic or bootstrap CIs for θ_k, $k=1,2$. So, we expect asymptotic approaches to fail, including the intersection methods. The DGP is as follows.

DGP I-d: $F_1 = \mathrm{SM}(1, 1.6971, 8.3679)$, $F_2 = \mathrm{SM}(0.4, 1.1, 1.1)$, $\rho \in \{0, \pm 0.5, \pm 0.99\}$ in (5.3), stability index of F_2 is $\beta_2 = 1.21$, overlap portion $\lambda \in \{0.1, 0.5, 0.9\}$.

The simulation results are provided in Table 25 for Gini index difference, Table 24 for mean difference, and Table 26 for LC ordinate difference at $p = 0.5$.

Table 15. DGP I-a: Coverage and width of 95% confidence intervals for the mean difference

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.5	0.1	100	99.9	99.9	95.5	94.7	97.7	0.1675	0.1746	0.1064	0.1071	0.1637
		200	99.9	99.9	94.2	93.9	97.9	0.1189	0.1215	0.0754	0.0759	0.1158
		500	99.8	99.8	94.7	94.6	97.8	0.0755	0.0762	0.0478	0.048	0.0732
		1000	99.5	99.5	94.7	94.9	97.9	0.0534	0.0536	0.0338	0.0339	0.0516
	0.5	100	99.5	99.6	94.8	94	97.8	0.1668	0.1739	0.1143	0.1157	0.1615
		200	99.5	99.3	94.8	94.2	97.6	0.1195	0.1222	0.0817	0.0822	0.1177
		500	99.5	99.7	94.5	94.1	97.1	0.0754	0.076	0.0514	0.0516	0.0732
		1000	99.8	99.9	96.4	96.3	97.9	0.0534	0.0538	0.0365	0.0366	0.0512
	0.9	100	99.3	99.4	96.8	96.1	96.9	0.1668	0.1732	0.1218	0.1235	0.16
		200	99.3	99.3	95	94.4	96.3	0.1188	0.1214	0.0866	0.0873	0.113
		500	99.3	99.1	94	93.6	95.1	0.0755	0.0762	0.055	0.0552	0.0721
		1000	99.1	98.9	94.7	94.2	96.8	0.0533	0.0534	0.0388	0.039	0.0506
0	0.1	100	99.8	99.9	94.1	93.9	98	0.1679	0.1751	0.1044	0.1056	0.1667
		200	100	99.9	95	94.3	97.9	0.119	0.1217	0.0739	0.0744	0.1153
		500	99.9	99.9	94.6	94.5	97.3	0.0753	0.0759	0.0467	0.0468	0.072
		1000	99.6	99.7	95.1	95.3	97.8	0.0535	0.0537	0.0331	0.0332	0.0515
	0.5	100	99.8	99.8	95.5	94.4	98.9	0.1659	0.1727	0.1031	0.104	0.1615
		200	99.8	99.7	96	95.1	98.4	0.1192	0.1217	0.074	0.0746	0.1171
		500	99.7	99.6	94.1	94.2	98.2	0.0756	0.0762	0.0468	0.047	0.0728
		1000	99.6	99.8	94	93.8	98.1	0.0536	0.0539	0.0332	0.0333	0.0515
	0.9	100	99.8	99.8	94.1	93.6	97.5	0.1677	0.175	0.104	0.1056	0.1619
		200	99.9	99.8	95.1	94.5	97.8	0.1188	0.1213	0.0737	0.0743	0.1137
		500	99.8	99.8	95.4	94.7	98	0.0755	0.076	0.0468	0.047	0.0739
		1000	99.9	99.8	93.6	93.2	97.2	0.0535	0.0537	0.0331	0.0333	0.0514
0.5	0.1	100	99.5	99.4	92.9	92.2	96.5	0.1664	0.173	0.1012	0.1023	0.1615
		200	99.9	99.9	95.2	94.9	98.1	0.1188	0.1212	0.072	0.0725	0.1143
		500	100	99.8	94.4	94.4	98.3	0.0756	0.0763	0.0458	0.046	0.0729
		1000	100	100	95.3	95.1	98.9	0.0535	0.0537	0.0324	0.0325	0.0519
	0.5	100	100	100	95	93.3	98.6	0.1686	0.1764	0.0917	0.0929	0.1652
		200	99.9	99.9	94.6	93.9	98.9	0.1189	0.1215	0.0646	0.065	0.1138
		500	100	100	93.9	93.2	99.3	0.0756	0.0762	0.041	0.0412	0.0736
		1000	100	100	95.8	94.9	99.1	0.0534	0.0536	0.0289	0.0291	0.0518
	0.9	100	100	100	96.2	95.6	99.7	0.1672	0.1746	0.079	0.0801	0.1632
		200	100	100	95.1	94.7	99.2	0.1188	0.1213	0.0562	0.0566	0.115
		500	100	100	94.8	95	98.9	0.0754	0.0761	0.0357	0.0358	0.0722
		1000	100	100	94.4	94.9	99.2	0.0534	0.0537	0.0252	0.0253	0.0505
0.99	0.1	100	99.9	99.9	94.8	92.9	99.2	0.1674	0.1742	0.0989	0.0999	0.1638
		200	100	100	95.1	95.1	98.3	0.1192	0.1216	0.0703	0.0707	0.116
		500	99.9	99.8	95.8	95.7	98.9	0.0755	0.076	0.0444	0.0446	0.0736
		1000	99.7	99.7	95.6	95.6	98.9	0.0534	0.0536	0.0314	0.0316	0.0522
	0.5	100	100	100	95.5	94.6	100	0.1662	0.1736	0.0742	0.0748	0.1612
		200	100	100	94.5	94.4	100	0.1185	0.1209	0.0527	0.0528	0.1103
		500	100	100	93.9	93.8	100	0.0755	0.0762	0.0336	0.0338	0.0718
		1000	100	100	93.9	94.6	100	0.0532	0.0533	0.0236	0.0237	0.0511
	0.9	100	100	100	93.8	91.7	100	0.1675	0.1746	0.0365	0.0352	0.1611
		200	100	100	94	92.7	100	0.1194	0.1221	0.0261	0.0259	0.1118
		500	100	100	94.5	93.7	100	0.0752	0.0758	0.0165	0.0165	0.07
		1000	100	100	95.4	94.7	100	0.0534	0.0536	0.0117	0.0118	0.05

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean differences by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-a for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 16. DGP I-a: Coverage and width of 95% confidence intervals for the Gini index difference

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.5	0.1	100	99.3	99.8	93.8	93.7	98	0.2045	0.227	0.1266	0.133	0.205
		200	99.7	99.8	93.9	93.6	97.3	0.1461	0.1554	0.0903	0.0929	0.1426
		500	100	100	95.9	95.7	98.4	0.0936	0.0965	0.0576	0.0585	0.0901
		1000	99.7	99.6	94	94.3	98.2	0.0664	0.0674	0.0408	0.0412	0.0641
	0.5	100	99.9	99.9	92.6	92.8	98.5	0.2037	0.2258	0.1218	0.1273	0.2043
		200	99.8	99.9	95	94.3	98.3	0.1468	0.156	0.0876	0.0899	0.147
		500	100	100	94.8	94.4	97.6	0.0936	0.0963	0.0557	0.0564	0.0912
		1000	99.9	99.9	95.3	95.2	98.4	0.0667	0.0678	0.0396	0.04	0.0648
	0.9	100	99.9	100	94.3	94.7	98.5	0.2011	0.2206	0.1166	0.1218	0.2
		200	100	100	95	94.9	99.3	0.1463	0.155	0.0842	0.0863	0.1443
		500	100	100	94.5	93.9	98.6	0.0937	0.0963	0.0537	0.0545	0.0903
		1000	99.8	99.8	94.5	94.2	98.3	0.0664	0.0674	0.038	0.0384	0.0646
0	0.1	100	99.6	99.9	93.8	93.6	97.8	0.203	0.2254	0.1265	0.1334	0.2027
		200	99.5	99.6	94.7	94.7	98.3	0.1461	0.1553	0.0909	0.0939	0.1432
		500	99.9	99.8	95.3	95.2	98.1	0.0932	0.0958	0.0577	0.0587	0.0909
		1000	99.9	99.9	95.2	94.6	98.7	0.0667	0.0678	0.0414	0.0418	0.0645
	0.5	100	99.6	99.7	92.9	93.4	97.8	0.2012	0.221	0.1254	0.132	0.2019
		200	99.7	99.7	94.4	94.4	97.9	0.1462	0.1551	0.091	0.0938	0.1445
		500	99.8	99.8	95	94.9	98.2	0.0937	0.0964	0.0581	0.0592	0.0899
		1000	100	100	94.4	94.4	98.2	0.067	0.0681	0.0415	0.0419	0.0647
	0.9	100	99.9	99.9	94	94.4	98.7	0.2039	0.2255	0.1265	0.134	0.2054
		200	99.5	99.6	95.2	95.4	97.7	0.1459	0.1545	0.0906	0.0936	0.1478
		500	99.9	99.8	94.6	94.4	98.4	0.0939	0.0966	0.0582	0.0592	0.0909
		1000	100	99.8	95.4	95.4	98.6	0.0666	0.0675	0.0412	0.0415	0.0649
0.5	0.1	100	99.9	99.9	93.6	93.6	98.8	0.2013	0.222	0.124	0.1302	0.2004
		200	99.8	99.8	95.3	94.7	98.1	0.1453	0.1537	0.0892	0.0919	0.1407
		500	99.8	99.8	95.4	95.4	98.3	0.0938	0.0964	0.0575	0.0585	0.091
		1000	100	100	95.3	94.4	98.6	0.0668	0.0678	0.0409	0.0413	0.065
	0.5	100	99.7	99.8	94	93.6	98.9	0.2045	0.2265	0.1199	0.1271	0.2053
		200	99.8	99.9	95.4	95	98.5	0.1455	0.1544	0.0853	0.0881	0.1425
		500	99.8	99.8	95.4	95.2	98.5	0.0937	0.0965	0.0548	0.0558	0.091
		1000	99.9	99.9	94.9	94.7	97.6	0.0665	0.0674	0.0388	0.0391	0.0632
	0.9	100	99.7	99.8	91.9	92.5	98.6	0.2024	0.224	0.1125	0.1196	0.1996
		200	100	100	92.7	92.7	99	0.1455	0.154	0.0808	0.0835	0.1419
		500	100	100	93.8	93.8	98.5	0.0936	0.0962	0.0519	0.0529	0.0904
		1000	99.9	99.9	95.8	96.1	99.3	0.0666	0.0676	0.0368	0.0373	0.0643
0.99	0.1	100	100	100	94.1	94.5	97.7	0.2024	0.2227	0.1201	0.1259	0.2025
		200	99.9	99.9	93.2	92.5	98.4	0.1461	0.1548	0.0865	0.089	0.1448
		500	99.9	99.9	95.4	94.7	98.8	0.0935	0.096	0.0552	0.056	0.0906
		1000	99.9	99.9	94.6	94.4	98.6	0.0666	0.0676	0.0393	0.0398	0.0645
	0.5	100	100	100	93.5	93.5	99.8	0.2019	0.2229	0.0926	0.0965	0.2012
		200	100	100	94	93.7	99.9	0.1457	0.154	0.0664	0.0676	0.1399
		500	100	100	94.7	94.5	100	0.0936	0.0963	0.0426	0.0433	0.0901
		1000	100	100	95.1	94.9	100	0.0663	0.0672	0.0301	0.0304	0.0623
	0.9	100	100	100	94.4	93.1	100	0.2039	0.225	0.0518	0.0521	0.1975
		200	100	100	94.5	93.2	99.9	0.1459	0.1546	0.0371	0.0378	0.1411
		500	100	100	93.8	92.2	100	0.0932	0.0958	0.0237	0.024	0.0886
		1000	100	100	94.3	93.8	99.8	0.0666	0.0676	0.017	0.0172	0.0642

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the Gini index differences by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-a for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 17. DGP I-a: Coverage and width of 95% confidence intervals for the difference in Lorenz curve at percentile 0.5

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.99	0.1	100	99.8	99.9	95.1	97.5	98.3	0.1412	0.1661	0.0846	0.0968	0.1382
		200	99.8	100	95.3	96.5	98.5	0.1004	0.1093	0.06	0.0645	0.0965
		500	100	100	96.1	96.4	98	0.0636	0.0659	0.038	0.0392	0.0608
		1000	100	100	94.3	94.7	97.4	0.0451	0.0459	0.0269	0.0274	0.0437
	0.5	100	100	100	93.9	96.8	99.8	0.142	0.1678	0.0706	0.084	0.1399
		200	100	100	95.3	96.9	99.6	0.1002	0.1096	0.0496	0.0546	0.0952
		500	100	100	95.7	96.7	99.4	0.0635	0.0658	0.0313	0.0327	0.0605
		1000	100	99.9	95.6	96.1	99.4	0.045	0.0459	0.0223	0.0228	0.0433
	0.9	100	100	100	94	98.8	99.9	0.1415	0.167	0.0513	0.069	0.1377
		200	100	100	94.5	96.6	99.8	0.1005	0.1097	0.0364	0.0427	0.0978
		500	100	100	95.1	96	99.6	0.0638	0.0663	0.0232	0.0249	0.0599
		1000	100	100	96.2	96.6	99.9	0.0451	0.0461	0.0164	0.0171	0.0425
-0.5	0.1	100	99.6	99.9	94.1	96.4	97.7	0.1424	0.1683	0.0877	0.0996	0.1402
		200	99.7	99.9	94.7	96.1	97.7	0.1005	0.1099	0.0618	0.0663	0.0957
		500	100	100	94.5	95.7	98.1	0.0637	0.066	0.0391	0.0402	0.0612
		1000	99.7	99.7	93.6	94.4	98.1	0.045	0.0458	0.0277	0.0281	0.0432
	0.5	100	99.9	100	93.3	96.6	98.3	0.1421	0.1675	0.0845	0.0966	0.14
		200	100	100	95.4	96.6	98.9	0.1009	0.1103	0.0599	0.0645	0.0987
		500	99.9	100	94.7	95.3	97.7	0.0636	0.0659	0.0378	0.0389	0.0618
		1000	100	100	95.2	95.4	98.5	0.0451	0.046	0.0267	0.0273	0.0434
	0.9	100	99.8	100	94	96.9	98.6	0.1413	0.1667	0.0813	0.0943	0.1373
		200	100	100	95.7	97.4	99.3	0.1006	0.1097	0.0576	0.0624	0.0966
		500	100	100	95.1	95.6	98.9	0.0637	0.0661	0.0363	0.0376	0.0613
		1000	99.8	99.9	94.9	95	98.4	0.045	0.0458	0.0257	0.0262	0.0437
0	0.1	100	99.9	99.9	94	97	98	0.1418	0.1673	0.0881	0.1003	0.1388
		200	99.5	99.6	94.8	96	97.8	0.1004	0.1097	0.0624	0.0669	0.0959
		500	99.8	99.9	95.1	95.2	98.5	0.0635	0.0658	0.0394	0.0406	0.0619
		1000	99.8	99.9	95.5	95.3	98.1	0.0451	0.046	0.0279	0.0284	0.0432
	0.5	100	99.6	100	93.9	97	98.3	0.1413	0.1661	0.0878	0.1001	0.1384
		200	99.5	99.8	94.4	96.2	98.1	0.1006	0.1099	0.0625	0.067	0.0971
		500	100	100	95.3	95.9	98.5	0.0637	0.066	0.0395	0.0407	0.0608
		1000	100	100	94.7	95.1	97.9	0.0452	0.046	0.028	0.0285	0.0435
	0.9	100	99.9	100	93.7	96.8	98.4	0.1425	0.168	0.0881	0.1007	0.1403
		200	99.6	99.9	96.2	96.8	97.8	0.1005	0.1096	0.0622	0.0668	0.0992
		500	100	100	94.4	94.9	98.6	0.0637	0.0661	0.0395	0.0407	0.0613
		1000	99.9	100	94.8	95.4	98.3	0.0451	0.0459	0.0279	0.0283	0.0436
0.5	0.1	100	100	100	95.1	97	97.7	0.1415	0.1666	0.087	0.0991	0.1375
		200	99.9	100	94.9	96.4	98.2	0.1001	0.1092	0.0614	0.0659	0.0935
		500	99.8	99.8	95.7	96.4	98.5	0.0637	0.0661	0.039	0.0402	0.0613
		1000	99.9	99.9	96	96	98.4	0.0451	0.046	0.0277	0.0281	0.0436
	0.5	100	99.8	100	94.2	97.2	98.1	0.1427	0.1683	0.0838	0.0962	0.1396
		200	99.9	100	96.3	97.1	98.6	0.1003	0.1097	0.0588	0.0633	0.0958
		500	99.8	99.8	96	96.6	98.7	0.0638	0.0661	0.0373	0.0385	0.0617
		1000	99.8	99.9	95.2	95.6	98	0.045	0.0459	0.0263	0.0269	0.0423
	0.9	100	99.9	100	93.8	97	98.4	0.1416	0.1671	0.079	0.0917	0.1361
		200	100	100	93.5	94.8	98.7	0.1001	0.1093	0.0557	0.0605	0.0952
		500	100	100	94.2	94.9	98.6	0.0637	0.0661	0.0354	0.0368	0.0615
		1000	100	100	95	95.6	99.4	0.0451	0.0459	0.025	0.0255	0.0431
0.99	0.1	100	100	100	94.9	98.1	98.3	0.1417	0.1669	0.0839	0.0963	0.1388
		200	100	100	93.9	95.2	98.6	0.1006	0.1096	0.0594	0.064	0.0971
		500	99.8	100	95.9	96.5	98.8	0.0637	0.0661	0.0376	0.0388	0.0611
		1000	99.9	99.9	95.3	95.1	98.7	0.045	0.0459	0.0266	0.0271	0.0434
	0.5	100	100	100	94.6	97.9	100	0.1411	0.1665	0.0647	0.0775	0.1381
		200	100	100	94.5	96.6	100	0.1005	0.1097	0.0458	0.0504	0.0934
		500	100	100	95	95.8	99.9	0.0637	0.066	0.029	0.0304	0.0605
		1000	100	100	94.5	95.2	99.9	0.045	0.0457	0.0204	0.0209	0.0421
	0.9	100	100	100	95.5	99.4	99.9	0.1425	0.1679	0.0364	0.0537	0.1347
		200	100	100	94.5	98	99.8	0.1003	0.1096	0.0255	0.0323	0.0943
		500	100	100	94.3	96.6	100	0.0636	0.0658	0.0161	0.018	0.0603
		1000	100	100	94.8	96.2	100	0.0451	0.0459	0.0115	0.0122	0.0433

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the difference in Lorenz curve at percentile 0.5 by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-a for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 18. DGP I-b: Coverage and width of 95% confidence intervals for the mean difference

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.5	0.1	100	99.4	99.6	94.3	93.3	97.2	0.2416	0.2698	0.1617	0.1735	0.2461
		200	99.6	99.5	94.2	94	97.9	0.176	0.1897	0.118	0.1246	0.1793
		500	99.5	99.5	94.2	94.8	96.4	0.1119	0.1167	0.0749	0.0775	0.1138
		1000	99.6	100	95.6	95.1	97.9	0.0802	0.082	0.0537	0.0548	0.0816
	0.5	100	99.3	99.3	93	92.8	97.4	0.2415	0.2711	0.1707	0.1831	0.2488
		200	98.9	98.9	93.3	93.3	96.3	0.1746	0.1879	0.1233	0.1293	0.178
		500	99.4	99.4	94.4	94	96.9	0.1136	0.1189	0.0802	0.0831	0.1167
		1000	99.6	99.6	95.9	95.9	96.5	0.0802	0.0825	0.0565	0.0578	0.0807
	0.9	100	98.9	99.1	95	94	95.9	0.2467	0.2798	0.1832	0.1974	0.249
		200	99	99.2	94.2	94.3	95	0.1758	0.1897	0.13	0.1364	0.1803
		500	99	99	95.4	95	96.7	0.113	0.1181	0.0835	0.086	0.1132
		1000	99.5	99.5	93.5	93.1	95.4	0.0809	0.0835	0.0597	0.0613	0.0809
0	0.1	100	99.5	99.8	94.6	94.3	96.9	0.244	0.2745	0.1611	0.1753	0.2469
		200	99.5	99.6	93.8	93.8	97.1	0.1731	0.1856	0.1139	0.12	0.176
		500	99.5	99.5	95.5	95.8	97.4	0.1126	0.1181	0.0745	0.0778	0.1168
		1000	99.3	99.1	94.1	93.9	96.2	0.0802	0.0821	0.053	0.0543	0.0801
	0.5	100	99.3	99.6	94.8	94.7	97.6	0.2425	0.2695	0.1596	0.1722	0.248
		200	99.5	99.5	94.9	94.6	97.7	0.1764	0.1914	0.1167	0.125	0.1788
		500	99	99.2	93.2	92.2	97	0.1115	0.1155	0.0735	0.0757	0.1115
		1000	99.7	99.8	95.2	94.3	97.2	0.0796	0.0815	0.0525	0.0537	0.0811
	0.9	100	99.2	99.5	92.8	92.1	95.8	0.2464	0.2788	0.1624	0.1782	0.2492
		200	99.5	99.8	93.9	93.4	97.2	0.1763	0.1931	0.1165	0.1255	0.1821
		500	99.6	99.5	93.9	93	96.7	0.1124	0.1168	0.0742	0.0769	0.1162
		1000	99.6	99.8	93.7	94.1	97.7	0.0803	0.0832	0.053	0.0553	0.083
0.5	0.1	100	99.6	99.6	93.1	93.5	97.2	0.2427	0.2745	0.1573	0.1732	0.2486
		200	99.6	99.8	94.5	94.5	98.4	0.1767	0.1938	0.1149	0.1251	0.1813
		500	99.6	99.8	94.6	94	97.6	0.1121	0.1165	0.0727	0.0754	0.1134
		1000	99.6	99.8	93.7	94.2	97.4	0.0801	0.0821	0.052	0.0534	0.082
	0.5	100	99.7	99.9	92.7	92.9	98.4	0.2438	0.2738	0.146	0.1617	0.248
		200	99.9	100	93.6	93.4	98.3	0.1741	0.1859	0.1042	0.1111	0.1751
		500	99.7	99.7	94.8	94.8	98.5	0.1131	0.1178	0.0683	0.0714	0.1135
		1000	99.9	99.8	95.4	94.6	97.8	0.0798	0.0819	0.0482	0.0499	0.0809
	0.9	100	99.8	100	93.4	93.2	98.5	0.2441	0.2745	0.1338	0.1509	0.2468
		200	99.8	99.8	92.9	92	98.1	0.1739	0.1865	0.0955	0.1033	0.1793
		500	99.9	100	95	94.4	98.2	0.113	0.118	0.0628	0.0664	0.1146
		1000	100	100	94.5	93.8	99	0.0805	0.083	0.045	0.047	0.0823
0.99	0.1	100	99.2	99.3	92.6	92.2	97.1	0.2423	0.2753	0.1536	0.1709	0.2463
		200	99.5	99.6	92.3	92	97.3	0.1737	0.1878	0.1101	0.1179	0.1743
		500	99.8	99.8	93.5	93.8	97.1	0.112	0.1166	0.0712	0.0742	0.1147
		1000	99.8	99.7	93.4	93	97.4	0.0807	0.083	0.0514	0.0531	0.0815
	0.5	100	100	100	92.7	92.8	99.2	0.2442	0.2784	0.1281	0.1498	0.2484
		200	99.9	99.9	92.4	93	99.4	0.1773	0.1922	0.0937	0.1043	0.184
		500	100	100	93.7	93.8	98.8	0.1133	0.119	0.0605	0.0654	0.1144
		1000	100	100	94.5	94.1	99	0.0806	0.083	0.0431	0.0454	0.0813
	0.9	100	100	100	89.5	92.1	98.8	0.2409	0.2678	0.0903	0.1159	0.2342
		200	100	100	90.9	93.4	99.5	0.1729	0.1846	0.0663	0.0785	0.1737
		500	100	100	93.8	94.2	99.6	0.1122	0.1162	0.0446	0.0497	0.1125
		1000	100	100	92.7	93.2	99.4	0.08	0.0821	0.0323	0.0354	0.0798

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean differences by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-b for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 19. DGP I-b: Coverage and width of 95% confidence intervals for the Gini index difference

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.5	0.1	100	99.5	99.8	93.2	93.7	97.6	0.2481	0.2955	0.1568	0.1783	0.2506
		200	99.6	99.6	94.5	95.1	98	0.1865	0.2142	0.1185	0.1318	0.1856
		500	99.1	99.1	92	92.3	97.4	0.1211	0.1319	0.0769	0.0824	0.12
		1000	99.3	99.4	94.6	94.2	98	0.0882	0.0936	0.0561	0.059	0.0879
	0.5	100	99.4	99.7	92.4	93.6	96.9	0.2483	0.2952	0.1527	0.1733	0.2517
		200	99.7	99.7	94	93.6	97.3	0.1843	0.2097	0.1134	0.1255	0.1846
		500	99.6	99.8	94.6	94.3	98.3	0.1236	0.136	0.0767	0.0833	0.1234
		1000	99.8	99.9	94.3	95.1	98.7	0.0884	0.0946	0.0549	0.0583	0.0869
	0.9	100	99.5	99.8	92.4	93.4	98	0.2511	0.3018	0.1497	0.1724	0.2549
		200	99.7	99.8	93.6	93.7	98.2	0.1853	0.2121	0.1111	0.1245	0.1836
		500	99.6	99.7	92.8	92.5	96.7	0.1226	0.1345	0.074	0.0803	0.1217
		1000	99.8	99.9	94.5	94.4	97.6	0.0892	0.0958	0.054	0.0579	0.088
0	0.1	100	99.6	99.9	92.3	93.7	98	0.2501	0.3006	0.1589	0.1832	0.2524
		200	99.1	99.4	91.2	91.9	97.7	0.1832	0.2083	0.1165	0.129	0.1843
		500	99.6	99.6	94.5	94.7	97.6	0.1221	0.134	0.0782	0.0849	0.1233
		1000	99.7	99.8	93.5	93.1	97.3	0.0882	0.0938	0.0565	0.0595	0.0875
	0.5	100	99.6	99.8	91.3	92.3	97.9	0.2476	0.2939	0.1568	0.1799	0.2531
		200	99.7	99.7	92.6	92.8	97.8	0.1857	0.2134	0.1185	0.1335	0.1851
		500	99.6	99.6	92	92.2	97.2	0.1207	0.1307	0.077	0.082	0.1201
		1000	99.8	99.8	94.5	94.1	98.4	0.0874	0.0925	0.0559	0.0586	0.0855
	0.9	100	99.5	99.9	91.7	93.2	98.1	0.2495	0.2993	0.1583	0.1833	0.2551
		200	99.4	99.6	91.9	92.1	97	0.1857	0.2121	0.1184	0.1338	0.1846
		500	99.8	99.8	93.9	94.3	98.1	0.1223	0.1337	0.0783	0.0841	0.1225
		1000	99.7	99.7	94.3	94	97.6	0.0881	0.0938	0.0565	0.0602	0.0875
0.5	0.1	100	99.7	99.8	91.2	92.2	97.7	0.2485	0.2961	0.1561	0.1788	0.2526
		200	99.6	99.8	92.5	93.1	97.3	0.1858	0.2129	0.1173	0.1314	0.1903
		500	99.9	99.9	93.8	93.9	97.8	0.1213	0.1321	0.0767	0.0826	0.1207
		1000	99.4	99.5	93.7	93.6	97.7	0.088	0.0936	0.0558	0.059	0.0857
	0.5	100	99.6	99.8	90.6	92.5	97.1	0.2496	0.2983	0.1499	0.1743	0.2513
		200	99.3	99.6	92.3	93.1	98	0.1838	0.2102	0.111	0.1243	0.1837
		500	99.4	99.3	92.1	92.1	97	0.1231	0.135	0.0754	0.082	0.1233
		1000	99.5	99.7	94.7	93.8	98.2	0.0878	0.0936	0.0537	0.0572	0.0878
	0.9	100	100	100	91.1	92.3	98	0.2485	0.2962	0.1432	0.1685	0.2499
		200	99.8	99.8	91.1	90.9	97.9	0.1834	0.208	0.1065	0.1201	0.1832
		500	100	100	93	93.1	98	0.1227	0.1341	0.072	0.0785	0.1227
		1000	99.9	99.9	94	92.6	98.1	0.0885	0.0944	0.0523	0.0558	0.0883
0.99	0.1	100	99.5	99.8	92.2	93.1	97.7	0.2486	0.2964	0.1508	0.1728	0.2495
		200	99.5	99.7	92.8	93.2	98.1	0.1818	0.2045	0.1105	0.1224	0.1799
		500	99.5	99.6	93.7	93.3	97.8	0.1215	0.1327	0.0745	0.0808	0.1213
		1000	99.7	99.7	95.3	95.2	98.5	0.0889	0.0952	0.0548	0.0585	0.0882
	0.5	100	100	100	91.8	93	99.6	0.2494	0.2978	0.1211	0.1447	0.2499
		200	100	100	93.4	93.4	99.4	0.1866	0.215	0.0922	0.1075	0.1886
		500	100	100	94.2	95	99.3	0.1236	0.1362	0.0623	0.0714	0.1227
		1000	100	100	94.2	94.3	99.5	0.0887	0.0952	0.0451	0.0496	0.0866
	0.9	100	100	100	84.5	88.3	99.4	0.2466	0.2931	0.0768	0.104	0.2453
		200	100	100	88.8	91	99.6	0.1826	0.2071	0.0604	0.0796	0.1804
		500	100	100	90.8	93.3	99.8	0.122	0.1327	0.0434	0.0533	0.1189
		1000	100	100	90.7	92.7	99.3	0.088	0.0936	0.0324	0.0387	0.0863

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the Gini index differences by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-b for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 20. DGP I-b: Coverage and width of 95% confidence intervals for the difference in Lorenz curve at percentile 0.5

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.99	0.1	100	99.8	100	95.1	97.6	98.4	0.162	0.19	0.0978	0.1116	0.1618
		200	99.8	99.9	94.7	95.7	97.6	0.1157	0.1272	0.0697	0.076	0.1119
		500	100	100	94.5	94.7	97.9	0.0738	0.0771	0.0444	0.0462	0.0724
		1000	99.8	99.8	95.2	94.9	99	0.0527	0.0541	0.0318	0.0326	0.0507
	0.5	100	100	100	93.9	96.9	99	0.1601	0.1868	0.0811	0.0948	0.1574
		200	100	100	94.3	95.3	99.3	0.1155	0.1264	0.0589	0.0649	0.1112
		500	100	100	94	94.3	99.4	0.0746	0.0784	0.0383	0.0409	0.0733
		1000	100	100	94.3	94.5	99	0.0527	0.0542	0.0271	0.028	0.0512
	0.9	100	100	100	93	98.1	99.6	0.1618	0.189	0.0634	0.0814	0.1601
		200	100	100	93.3	95.8	99.5	0.1157	0.1271	0.0461	0.0541	0.11
		500	100	100	94	95.4	99.7	0.0744	0.0782	0.0302	0.0331	0.0724
		1000	100	100	93.9	93.9	99.9	0.0533	0.0551	0.0221	0.0237	0.0508
-0.5	0.1	100	99.9	100	94.6	97.2	98.1	0.1606	0.1876	0.0994	0.1121	0.158
		200	99.8	99.9	95	95.9	98	0.1161	0.1276	0.0719	0.0774	0.111
		500	99.9	99.9	93.7	93.8	97.9	0.074	0.0774	0.0457	0.0476	0.0717
		1000	99.8	99.8	94.1	93.8	98.4	0.0529	0.0544	0.0327	0.0334	0.0512
	0.5	100	99.7	100	93.3	95.9	97.7	0.1611	0.1879	0.0963	0.1096	0.1593
		200	99.8	100	95.4	96.8	98.3	0.1155	0.1267	0.069	0.0746	0.1117
		500	99.8	99.8	94.9	95.4	98.7	0.0747	0.0786	0.0447	0.0467	0.0729
		1000	99.9	99.9	95	95.3	98.6	0.053	0.0545	0.0317	0.0326	0.0509
	0.9	100	99.8	100	94.6	96.6	98.7	0.1617	0.1893	0.0929	0.107	0.1602
		200	99.7	99.8	95.3	96.6	98.4	0.1159	0.1275	0.0668	0.0728	0.11
		500	99.7	99.9	93.8	94.6	98.3	0.0745	0.0783	0.043	0.0449	0.0719
		1000	100	100	94.8	94.8	98.5	0.0532	0.055	0.0308	0.0317	0.051
0	0.1	100	99.7	100	94.3	96.6	98	0.1612	0.1886	0.1005	0.114	0.1582
		200	99.4	99.4	93.6	95	98.3	0.1153	0.1265	0.0718	0.0772	0.1123
		500	99.8	99.8	95.5	95.7	97.9	0.0743	0.078	0.0463	0.0483	0.0738
		1000	99.8	99.8	94.7	94.6	97.3	0.0529	0.0545	0.033	0.0339	0.0509
	0.5	100	99.7	99.9	93.3	95.7	98.4	0.1608	0.1876	0.0999	0.1133	0.1621
		200	99.8	99.8	94.5	95.7	97.4	0.1159	0.1275	0.0722	0.0782	0.1114
		500	99.9	99.9	93.3	93.2	97.2	0.0738	0.0771	0.0459	0.0476	0.0723
		1000	99.9	99.9	96	96.2	98	0.0526	0.0541	0.0328	0.0336	0.0503
	0.9	100	99.8	100	93.7	96.6	98.7	0.1614	0.1889	0.1004	0.1143	0.1617
		200	99.9	100	93.8	95	97.8	0.116	0.1275	0.0723	0.0784	0.1106
		500	99.9	100	95.3	96.2	98.1	0.0743	0.078	0.0463	0.0482	0.0724
		1000	99.7	99.8	95.5	95.9	99	0.0529	0.0546	0.033	0.0341	0.0514
0.5	0.1	100	99.9	100	95	96.7	98.4	0.1611	0.1885	0.0994	0.1127	0.1598
		200	100	100	95	95.8	98	0.116	0.1274	0.0716	0.0776	0.114
		500	100	100	95	94.7	98.1	0.0741	0.0776	0.0457	0.0475	0.0724
		1000	100	99.9	94	94.4	97.9	0.0528	0.0543	0.0326	0.0334	0.0502
	0.5	100	99.7	100	93.3	95.9	98	0.1615	0.1888	0.0952	0.1087	0.1585
		200	99.8	100	94.6	95.6	98.3	0.1153	0.1264	0.0679	0.0734	0.1116
		500	99.6	99.6	93.4	94	97.5	0.0746	0.0783	0.0442	0.0461	0.0729
		1000	99.8	99.9	94.9	94.8	98.4	0.0528	0.0544	0.0312	0.0322	0.0511
	0.9	100	100	100	93.9	97.5	98.5	0.1611	0.188	0.0906	0.1045	0.1589
		200	100	100	92.5	93.7	98.1	0.1153	0.1263	0.065	0.0707	0.1107
		500	100	100	95.5	95.7	98.8	0.0744	0.0779	0.0421	0.044	0.0734
		1000	100	100	94.5	94.4	98.8	0.053	0.0547	0.0301	0.031	0.0514
0.99	0.1	100	99.7	99.8	94.6	96	98.5	0.1609	0.188	0.0956	0.1089	0.1572
		200	99.7	100	94.7	95.6	98.8	0.1147	0.1255	0.0681	0.0738	0.1092
		500	99.7	99.8	94.1	94.7	98.9	0.0741	0.0777	0.0441	0.0461	0.0719
		1000	99.8	99.8	95.3	96	98.9	0.0531	0.0549	0.0317	0.0327	0.0511
	0.5	100	100	100	95.1	97.5	99.8	0.1612	0.1882	0.075	0.0893	0.1575
		200	100	100	95.7	96.6	99.8	0.1163	0.1277	0.0545	0.0607	0.1126
		500	100	100	95.4	96.5	99.9	0.0747	0.0786	0.0352	0.038	0.072
		1000	100	100	94.5	94.5	100	0.0531	0.0548	0.0252	0.0263	0.0507
	0.9	100	100	100	91.6	98.2	99.9	0.1604	0.1872	0.0443	0.0627	0.1554
		200	100	100	93.2	96.5	99.9	0.115	0.1258	0.0327	0.041	0.1094
		500	100	100	94.6	96.3	100	0.0742	0.0775	0.022	0.025	0.0708
		1000	100	100	92.9	94.7	100	0.0528	0.0544	0.016	0.0177	0.0503

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the difference in Lorenz curve at percentile 0.5 by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-b for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 21. DGP I-c: Coverage and width of 95% confidence intervals for the mean difference

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.5	0.1	100	96.4	97.9	88.7	90.5	93.8	0.433	0.7051	0.3214	0.5186	0.4994
		200	96.2	97.5	88.5	90.7	93.5	0.3145	0.4315	0.2339	0.3251	0.3573
		500	97.7	98.2	92.4	92.9	94.8	0.2154	0.2745	0.1618	0.2085	0.2458
		1000	96.7	98.1	91.5	92.4	94.8	0.1598	0.2001	0.1207	0.1539	0.1819
	0.5	100	95.3	96.9	89.1	91.5	93.7	0.4292	0.6956	0.327	0.504	0.4871
		200	97.2	98.2	91.4	92.4	94.6	0.3174	0.4189	0.2421	0.3119	0.3612
		500	98	98.6	90.5	92.7	93.5	0.2117	0.285	0.1621	0.2189	0.2357
		1000	98	98.6	92.1	92.6	94	0.1561	0.1858	0.1198	0.1431	0.1789
	0.9	100	94.9	96.8	89.7	91.2	92.6	0.4229	0.6399	0.3294	0.4727	0.4793
		200	95.6	97	89.7	90.1	93.3	0.3238	0.44	0.253	0.3324	0.3714
		500	96.9	97.3	90.9	92.1	93.8	0.2203	0.3162	0.1727	0.244	0.2592
		1000	97.8	98.4	92.4	93.4	96.1	0.1565	0.1967	0.1224	0.153	0.1816
0	0.1	100	96.4	97.8	88.4	89.9	94.8	0.4484	1.0732	0.3324	0.8214	0.5168
		200	96.3	98.1	89.3	92	93.2	0.3161	0.4342	0.2333	0.3255	0.3591
		500	97.1	98.7	90.4	91.8	94.2	0.2107	0.2539	0.1565	0.1894	0.2366
		1000	98.1	99.2	91.8	92.4	95.2	0.1608	0.278	0.1209	0.2242	0.1879
	0.5	100	96.6	98.4	87.9	91.4	94.7	0.4064	0.5457	0.2957	0.3909	0.4629
		200	96.9	98.6	88.4	90.3	93.5	0.311	0.4132	0.2292	0.3085	0.3536
		500	97.4	98.3	91.9	93.4	96.4	0.2131	0.2613	0.1586	0.1974	0.2465
		1000	97.3	98.2	91.7	92.4	95	0.1556	0.1865	0.1164	0.1426	0.1787
	0.9	100	96.4	98	88.9	91.8	93.2	0.4237	0.6377	0.3107	0.4639	0.4827
		200	96.6	98.3	88.6	90.8	94.7	0.3152	0.4378	0.2327	0.3272	0.3559
		500	97.7	98	91.2	92.9	96.3	0.2136	0.2723	0.1591	0.2067	0.2469
		1000	97.9	98.9	92.5	93.5	95.8	0.1602	0.2131	0.1204	0.1671	0.1875
0.5	0.1	100	96.7	98.5	88.5	90	93.5	0.425	0.6188	0.309	0.4463	0.4819
		200	96.6	97.8	88.4	92.1	94.8	0.3357	1.3655	0.2483	1.1652	0.3922
		500	97.6	98.5	90.5	93.8	95	0.2172	0.2837	0.161	0.217	0.2558
		1000	97.8	98	91.7	92.1	94.5	0.1603	0.2094	0.1195	0.1634	0.1835
	0.5	100	96.5	98.1	84.7	88.7	93.6	0.4304	1.0659	0.3027	0.8637	0.4984
		200	98	98.9	90.7	92.2	94.6	0.3283	0.4725	0.2341	0.3576	0.3735
		500	98.1	98.6	91.8	92.8	95.4	0.2165	0.2857	0.1557	0.2161	0.2532
		1000	97.7	98.3	92	92.4	95	0.1565	0.1834	0.1131	0.1374	0.1815
	0.9	100	98.1	99.4	87.5	90.7	96.4	0.4124	0.5954	0.275	0.4285	0.4714
		200	96.8	98.2	88.3	91.2	96.3	0.3174	0.4429	0.2171	0.3311	0.3648
		500	98.6	99.4	89.7	92.3	96	0.212	0.2865	0.1474	0.2197	0.2521
		1000	98.1	98.5	92	93.5	94.6	0.1635	0.2847	0.116	0.2388	0.1884
0.99	0.1	100	96.1	97.8	85.7	89.5	92.2	0.4425	0.8479	0.321	0.6519	0.5126
		200	97	98.3	90	91.2	95.4	0.323	0.515	0.2349	0.3904	0.3738
		500	97.9	98.4	90	93.2	94.6	0.2102	0.2756	0.1534	0.2101	0.2395
		1000	98	99	90.2	92.2	95.4	0.1521	0.1837	0.1114	0.1395	0.1727
	0.5	100	97.2	98.4	85.6	90.2	95.8	0.4182	0.5877	0.2737	0.4202	0.4834
		200	98	98.5	87.6	90.4	95.2	0.3157	0.4102	0.2109	0.2985	0.3605
		500	98.8	99.2	90.4	92.3	96.2	0.211	0.2573	0.1435	0.1874	0.2403
		1000	98.6	98.8	90.1	92.9	95.7	0.1557	0.1831	0.1073	0.1338	0.1784
	0.9	100	99	99.5	84.2	90.9	96.4	0.4604	1.0876	0.2824	0.9797	0.5348
		200	98.6	99.3	86.3	91.6	95.7	0.3179	0.4478	0.194	0.3519	0.3628
		500	98.6	99.5	87.3	92.3	94.9	0.2163	0.3129	0.1371	0.2467	0.2452
		1000	98.8	99.2	89.2	91.7	95.8	0.1555	0.1829	0.0996	0.1312	0.1794

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean differences by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-c for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 22. DGP I-c: Coverage and width of 95% confidence intervals for the Gini index difference

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.5	0.1	100	97.6	98.3	84.6	88.5	95.8	0.2938	0.3789	0.1914	0.2517	0.3138
		200	96.8	97.7	87.3	90.2	94.5	0.2283	0.2859	0.1508	0.1909	0.2377
		500	98.2	99.2	90.5	92.5	95.7	0.1633	0.1993	0.1104	0.1388	0.169
		1000	98.6	99	90.7	92.2	96.3	0.1241	0.1486	0.085	0.1051	0.1292
	0.5	100	97.8	98.8	85.9	89.8	95.8	0.2921	0.3792	0.1863	0.2411	0.3113
		200	98.4	98.9	88.9	91.5	95.9	0.2305	0.2914	0.15	0.1936	0.2386
		500	98	98.9	88.6	91.4	95.2	0.1591	0.1906	0.1052	0.1306	0.1607
		1000	97.9	98.5	88.8	90	94.6	0.1223	0.1459	0.0823	0.1018	0.1253
	0.9	100	97.7	98.4	85.5	89.6	95.1	0.2929	0.3812	0.1832	0.2409	0.3065
		200	98.7	99.4	87.4	91.4	95.6	0.2338	0.2963	0.1499	0.1993	0.2388
		500	98.5	99.4	89.2	91.7	95.8	0.1638	0.201	0.1074	0.1374	0.1696
		1000	98.8	99.1	89.1	91.6	96.7	0.122	0.145	0.0808	0.0992	0.1279
0	0.1	100	97.2	98.7	85.7	89.7	96	0.2944	0.3847	0.1924	0.2555	0.3127
		200	97.8	98.4	87.5	90.1	94.4	0.2291	0.2878	0.152	0.1941	0.2362
		500	97.9	98.7	88.8	90.7	95.7	0.1614	0.1954	0.1091	0.1348	0.1695
		1000	97.8	98.6	89.4	91.4	95	0.1219	0.1447	0.0835	0.1026	0.1268
	0.5	100	97.7	99	85.1	89.2	96.3	0.2892	0.3723	0.1879	0.2432	0.3069
		200	98.2	98.9	87.3	90.3	94.3	0.2261	0.2811	0.1496	0.1918	0.2287
		500	97.6	98.5	89.1	91.5	95.4	0.1633	0.2003	0.1107	0.1381	0.1685
		1000	98.2	99	90.1	92.3	96	0.1221	0.1456	0.0836	0.102	0.1265
	0.9	100	97	97.9	83.8	87.9	93.9	0.2929	0.3803	0.1909	0.2547	0.3069
		200	97.7	98.5	87.1	90.5	95.5	0.2271	0.2833	0.1506	0.1936	0.2372
		500	98.4	98.9	89.7	93.5	96.7	0.1625	0.1984	0.1101	0.1372	0.1703
		1000	98.7	99	90.5	93	95.9	0.1233	0.1467	0.0846	0.1047	0.1308
0.5	0.1	100	98	99	85.9	90.3	94.9	0.2932	0.3823	0.1896	0.2492	0.3081
		200	98.8	99.7	88.6	90.9	96.4	0.2282	0.2851	0.1497	0.1911	0.2365
		500	97.7	98.5	91.1	93	95	0.1635	0.2001	0.1102	0.1388	0.1714
		1000	97.9	98.5	91.1	92.1	95.9	0.1234	0.1463	0.0842	0.1039	0.1275
	0.5	100	97.3	98.3	83.5	88.2	94.6	0.2856	0.3668	0.1771	0.2295	0.2978
		200	97.6	98.7	86.8	90.2	95.9	0.2322	0.2937	0.1489	0.1998	0.2396
		500	98.5	99.3	89.2	91.7	95.9	0.1635	0.2007	0.1075	0.1377	0.1698
		1000	98.5	98.7	91.2	92.3	95.9	0.1235	0.148	0.0824	0.1021	0.1311
	0.9	100	97.9	99	85.1	89.8	96.7	0.2899	0.3751	0.1738	0.2327	0.3039
		200	98	98.4	85.8	90.5	95.6	0.228	0.2855	0.1417	0.1877	0.2382
		500	99	99.2	87.4	91.3	96.2	0.1609	0.1953	0.1029	0.1302	0.1717
		1000	99.2	99.7	91.1	93.2	96.8	0.1237	0.1476	0.0809	0.1019	0.1296
0.99	0.1	100	97.6	99.2	85.3	89.1	95.5	0.2922	0.381	0.1838	0.2452	0.3094
		200	98.7	99.2	89	92.2	96.1	0.2313	0.2935	0.1485	0.1943	0.2458
		500	98.2	98.7	87.6	90.8	96.1	0.1594	0.1917	0.1043	0.1297	0.1664
		1000	98.6	99.3	89.3	91.7	96.2	0.1193	0.1401	0.079	0.0946	0.1248
	0.5	100	98.7	99.1	82.2	86.5	97.3	0.2924	0.3798	0.1536	0.2177	0.3115
		200	98.7	99.2	84.4	87.8	97	0.2295	0.2895	0.126	0.1751	0.2345
		500	99.1	99.5	88.1	91.8	97.3	0.1618	0.197	0.0934	0.1245	0.1688
		1000	99.2	99.1	89.2	92	97.6	0.1224	0.1456	0.073	0.0959	0.1278
	0.9	100	99.7	99.8	75.3	85.5	98.3	0.3011	0.3968	0.1218	0.2241	0.3114
		200	99.8	100	77.8	88.7	97.4	0.2288	0.2879	0.101	0.1685	0.2325
		500	99.7	99.8	82.8	88.7	98.4	0.1624	0.1973	0.0799	0.1255	0.1664
		1000	99.7	99.7	81.9	88	95.8	0.1219	0.1455	0.0632	0.0929	0.1262

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the Gini index differences by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-c for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 23. DGP I-c: Coverage and width of 95% confidence intervals for the difference in Lorenz curve at percentile 0.5

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.99	0.1	100	99.7	99.9	92.5	94.8	97.6	0.1678	0.1967	0.1022	0.1189	0.1722
		200	99.8	99.9	94.5	95.9	97.7	0.1233	0.1381	0.0753	0.0846	0.1199
		500	99.4	99.6	93.4	94.2	98.3	0.0824	0.0901	0.0507	0.056	0.0818
		1000	99.7	99.6	94.9	95.5	97.6	0.0602	0.065	0.0374	0.0408	0.0582
	0.5	100	99.7	100	92	95.9	99.2	0.1689	0.1988	0.0894	0.1083	0.1729
		200	99.9	99.9	89.9	91.9	99	0.1242	0.1396	0.0666	0.0776	0.1219
		500	99.9	99.9	92.6	93.6	99.1	0.0828	0.0908	0.0454	0.0521	0.0808
		1000	99.9	99.9	91.9	92.6	98.4	0.0607	0.0655	0.0339	0.039	0.0595
	0.9	100	99.9	100	89.6	94.6	99.2	0.1697	0.1998	0.0737	0.0964	0.1719
		200	99.9	100	90.2	93.2	98.9	0.1231	0.1382	0.0549	0.0664	0.1189
		500	100	100	92.7	94	99.5	0.0818	0.0889	0.0383	0.046	0.0782
		1000	100	100	91.5	93.4	99	0.0602	0.065	0.0292	0.035	0.0594
-0.5	0.1	100	99.7	100	92.2	94.2	97.7	0.1689	0.198	0.1052	0.1222	0.1758
		200	99.3	99.6	92.3	93.6	98	0.1233	0.138	0.0769	0.0856	0.1214
		500	99.9	99.9	93.8	94.7	98	0.0824	0.0901	0.0519	0.0571	0.0815
		1000	99.9	99.8	93.9	94.3	98.1	0.0604	0.0653	0.0383	0.0419	0.059
	0.5	100	99.7	99.9	94.1	96	98.1	0.1688	0.1982	0.1017	0.1182	0.1748
		200	99.6	100	93.1	94.2	97.3	0.124	0.1394	0.0752	0.0843	0.1215
		500	99.5	99.5	93.7	94.3	98.1	0.0813	0.0882	0.0497	0.0545	0.0784
		1000	99.5	99.7	91.9	91.8	97.7	0.0599	0.0647	0.0369	0.0402	0.0577
	0.9	100	99.8	99.9	93.3	96.1	97.7	0.1685	0.198	0.0983	0.1153	0.17
		200	99.9	100	93.8	95.7	98.1	0.1249	0.1406	0.0736	0.084	0.1192
		500	99.9	100	93.3	94.5	98.4	0.0826	0.0906	0.0493	0.0551	0.0811
		1000	99.9	99.9	93	93	98.8	0.0598	0.0644	0.0358	0.0393	0.0588
0	0.1	100	99.3	99.8	91.6	94.2	98.2	0.1686	0.1975	0.1055	0.123	0.1741
		200	99.7	99.9	92.2	93.2	97.7	0.1238	0.139	0.0778	0.0869	0.1199
		500	99.4	99.5	92.9	94.1	98.1	0.0819	0.0891	0.0518	0.0562	0.0817
		1000	99.4	99.7	92.7	93.8	96.9	0.0598	0.0642	0.038	0.0415	0.0582
	0.5	100	99.7	100	93	95.5	98.3	0.168	0.1966	0.1049	0.1206	0.1734
		200	99.9	99.9	93.1	94.5	97.5	0.1228	0.1371	0.0771	0.0862	0.1165
		500	99.5	99.7	92.5	92.8	97.1	0.0824	0.0901	0.0522	0.057	0.0812
		1000	99.6	99.6	92.9	93.6	97.2	0.0598	0.0645	0.0381	0.0413	0.0583
	0.9	100	99.5	99.7	91.9	94.1	96.6	0.169	0.1986	0.1056	0.1234	0.1718
		200	99.7	100	93.2	95	97.9	0.1231	0.1378	0.0774	0.087	0.1201
		500	99.9	99.9	93.5	94.5	98.2	0.0821	0.0897	0.052	0.0571	0.0819
		1000	99.3	99.5	93.9	94.4	97.6	0.0602	0.0648	0.0384	0.0423	0.0599
0.5	0.1	100	99.6	99.9	93.3	96.1	98	0.1688	0.1982	0.1046	0.121	0.1725
		200	99.7	99.8	94.2	95.4	98.1	0.1233	0.1381	0.0766	0.0854	0.1201
		500	99.4	99.4	94.6	95	97.8	0.0825	0.0901	0.0518	0.0573	0.0823
		1000	99.4	99.5	94.5	95.3	97.2	0.0603	0.065	0.0381	0.0422	0.0584
	0.5	100	99.1	99.6	92.5	94.8	98	0.1664	0.1948	0.0988	0.1142	0.1688
		200	99.9	99.9	92.3	94	97.3	0.1244	0.1402	0.0747	0.0852	0.1215
		500	99.5	99.7	92.4	93.8	98	0.0824	0.0901	0.0499	0.0555	0.081
		1000	99.5	99.5	93.7	93.9	97.8	0.0602	0.0648	0.0367	0.04	0.0596
	0.9	100	99.6	100	92.4	95.1	97.9	0.1677	0.1965	0.0953	0.1117	0.1705
		200	99.8	99.8	92.7	94.1	98.9	0.1233	0.1383	0.071	0.0808	0.1208
		500	99.9	100	93.4	94.3	98.9	0.0817	0.0889	0.0477	0.0528	0.0819
		1000	99.7	99.8	94	94.1	98.4	0.0603	0.0651	0.0355	0.0398	0.0596
0.99	0.1	100	99.7	100	92.2	94.3	97.9	0.1677	0.1974	0.1004	0.1174	0.1728
		200	99.9	100	93.5	94.8	98.2	0.1244	0.1404	0.0748	0.0845	0.1227
		500	99.8	100	92.5	93.8	98.3	0.0815	0.0883	0.0493	0.0542	0.0808
		1000	99.8	99.8	93.7	94.1	98	0.059	0.063	0.0359	0.0388	0.0577
	0.5	100	100	100	91.8	95	99.8	0.1683	0.1975	0.0801	0.0972	0.1754
		200	100	100	92.4	94.3	99.2	0.1238	0.1391	0.0601	0.0704	0.118
		500	99.8	99.8	92.1	93.1	99.6	0.082	0.0893	0.0409	0.0468	0.0817
		1000	99.9	99.9	94.3	94.4	99.3	0.0598	0.0645	0.0305	0.0346	0.0584
	0.9	100	99.9	100	86.5	92.7	99.9	0.1711	0.2017	0.0541	0.0815	0.1727
		200	100	100	86.3	92	99.5	0.1236	0.1387	0.0412	0.0568	0.118
		500	99.9	100	89.2	91.8	99.8	0.0821	0.0895	0.0302	0.0406	0.0797
		1000	100	100	86.2	88.8	98.7	0.0598	0.0644	0.0231	0.0296	0.0578

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the difference in Lorenz curve at percentile 0.5 by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-c for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 24. DGP I-d: Coverage and width of 95% confidence intervals for the mean difference

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.99	0.1	100	51.7	73.3	46.7	65.9	60.8	5.2741	539.0674	4.5493	457.5541	7.3195
		200	49.9	71.4	45.5	63	58.6	4.9477	974.4694	4.2813	868.9222	6.8601
		500	49.5	70.4	45.7	64.6	58.2	2.7768	74.7105	2.399	67.0266	3.8361
		1000	49.8	69.7	46.2	63.5	58.2	3.49	1330.778	3.0309	1259.153	4.8278
	0.5	100	50.2	72.5	45.8	65.6	57.7	3.7212	71.4542	3.203	59.4583	5.0873
		200	53.5	72.6	49.5	66.1	61.3	3.5381	71.3807	3.0557	62.0679	4.9045
		500	51.9	72.1	46.1	65.8	59.6	3.5986	628.833	3.1212	571.3023	4.9634
		1000	49.6	68	44	62.3	58.3	2.059	17.3476	1.7818	15.3628	2.8103
	0.9	100	48.8	72.5	44.2	64.7	56.1	3.033	30.5591	2.6104	23.9882	4.0938
		200	52.5	72.7	47.5	64.6	58.8	3.2229	66.2909	2.7857	50.0351	4.3778
		500	50.8	71	46.4	64.4	59.8	2.9252	62.3381	2.5351	53.4803	4.0364
		1000	52.4	71.6	48.2	67.3	61.3	2.5306	282.8267	2.1957	233.6607	3.5353
-0.5	0.1	100	49.6	72.5	43.9	65.4	57.4	3.0898	47.2424	2.6371	41.0526	4.1976
		200	50.1	73	45.7	65.6	57.9	3.1355	51.2639	2.6951	44.5162	4.2555
		500	50.6	69.2	46.8	63.9	61.2	4.2243	914.5841	3.664	861.6024	5.9074
		1000	52.2	71.7	48.3	66.1	61	3.1127	93.1743	2.7006	78.6868	4.3474
	0.5	100	50	73	45.7	66	58	3.5396	73.5865	3.0372	62.2823	4.8417
		200	50.9	72.1	47.3	65.3	58.8	2.7945	25.7138	2.4013	22.2143	3.8108
		500	53	73.7	48.8	67.9	61.6	2.5339	22.9785	2.1881	19.3846	3.4183
		1000	52	72.6	46.5	65.7	61.2	2.356	30.0117	2.0403	26.4763	3.2519
	0.9	100	46.6	69.4	42	61.8	54.3	4.1274	253.0313	3.5574	229.3338	5.6876
		200	50	68.8	46	62.5	56.4	4.0067	385.0814	3.4649	329.8669	5.5481
		500	51.2	70.3	47	63.3	60	2.5808	45.7789	2.231	37.8091	3.5765
		1000	50.8	70.4	46.3	64.7	60.2	2.8071	118.9625	2.4358	93.4466	3.8722
0	0.1	100	49.6	70.8	44.3	64.3	57.7	3.9423	635.7697	3.3801	590.3841	5.4142
		200	50.8	70.2	46	63.3	58.2	3.065	39.3917	2.632	34.151	4.1853
		500	51	66.5	46.5	62.1	57	2.8215	105.1333	2.4365	90.9014	3.8915
		1000	51	69	47.1	63.6	58.6	3.1445	210.3651	2.7279	124.0291	4.3365
	0.5	100	50.2	72.2	44.5	65.9	57.2	3.5904	92.1082	3.0723	70.8839	4.9494
		200	49.9	68.8	44.9	63.8	57.5	3.2305	59.0391	2.7768	50.4963	4.4297
		500	52.2	72.6	48.2	66.7	60.6	2.7206	42.567	2.3482	33.145	3.7246
		1000	54.3	70.7	49.8	65.8	62.3	2.6702	55.2656	2.3131	48.4164	3.6905
	0.9	100	51.5	75	45.8	67.5	58.4	4.3552	266.2451	3.741	194.995	6.0187
		200	53.4	72.7	48.9	66.8	59.7	7.1616	3307.878	6.2138	2916.877	9.9909
		500	52.4	70.9	47.1	65.1	59.3	2.472	19.6131	2.1309	17.0534	3.3887
		1000	52.7	71.4	48	64.6	59.9	2.6224	58.9787	2.2712	54.389	3.5998
0.5	0.1	100	52.4	73	48.4	67.7	57.9	4.6826	467.9815	4.0244	403.1008	6.4754
		200	52.1	72.2	46.8	66.9	60.4	6.516	1684.842	5.6478	1477.743	9.114
		500	50.6	69.5	46.2	64	57.9	2.1628	12.9075	1.8594	11.1947	2.9394
		1000	50.1	67.5	46.3	62.4	57.3	2.0387	17.7422	1.7602	16.0593	2.8103
	0.5	100	48	69.2	42.4	63.4	53.2	6.3405	2678.611	5.465	2478.119	8.926
		200	50.9	70.7	44.9	65.4	58.9	2.8025	48.2274	2.3947	45.294	3.8183
		500	52	72.4	47.2	65.2	60.7	2.7506	48.9232	2.3701	39.2784	3.7492
		1000	50.7	67.7	44.9	63	58.8	1.7999	8.6226	1.5495	7.2372	2.4652
	0.9	100	46.5	70.2	42.7	62.6	54	3.0577	37.552	2.586	32.2001	4.1185
		200	51.6	71	46.3	64.9	57.5	3.6572	105.0845	3.1369	82.4732	5.0346
		500	52	72.3	47.2	65.9	60.9	2.3621	18.1942	2.0277	15.7853	3.2437
		1000	51.6	70.7	47.1	65.3	61.1	2.7731	108.9459	2.3989	100.2385	3.8364
0.99	0.1	100	48.9	71.6	44.4	64.7	56.7	3.3068	53.787	2.8181	47.5833	4.4903
		200	50.1	70.8	44	63.7	57.7	3.3707	93.5103	2.8951	79.6189	4.6271
		500	53.3	70.7	48	65.4	58.8	3.205	90.5905	2.7694	80.7182	4.4576
		1000	50.9	69.6	45.8	63.8	58.7	2.446	36.7585	2.1155	32.2551	3.3511
	0.5	100	48.5	71.3	43.4	63.7	55	3.2066	42.3045	2.7084	40.0704	4.3553
		200	51.1	70.9	45.4	64.4	58.8	4.017	1014.765	3.4464	645.8075	5.5553
		500	53.8	70.5	48.4	66.6	61.1	2.4645	17.6792	2.1144	16.252	3.3915
		1000	49	69.8	43.6	63.8	57.7	2.1171	19.9415	1.8232	18.2375	2.8907
	0.9	100	49	70.5	43.3	63.8	56.2	3.3345	45.1225	2.8023	44.0161	4.5605
		200	48.9	69.8	43.2	63.2	57.1	2.9603	33.1894	2.5104	31.6593	4.0301
		500	54.8	73.7	49.8	67.9	62.5	3.9335	866.0394	3.3925	862.0195	5.4726
		1000	53.4	69.9	47.8	64.5	60.4	2.6466	42.4483	2.2823	40.3715	3.638

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean differences by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-d for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 25. DGP I-d: Coverage and width of 95% confidence intervals for the Gini index difference

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.99	0.1	100	69.1	81.7	47.5	61.4	69.5	0.2964	0.4245	0.1955	0.3092	0.3702
		200	68.8	82	48.2	65.2	66.6	0.2528	0.3627	0.1722	0.2783	0.3081
		500	65.8	80.6	48.2	65.7	65	0.2035	0.2787	0.145	0.2387	0.24
		1000	66.3	79.6	49.6	65.4	65.4	0.1718	0.2291	0.1258	0.2046	0.1986
	0.5	100	68.2	79.1	45.7	61.9	67.6	0.2986	0.4297	0.1921	0.3005	0.3725
		200	68.8	81.6	47.5	64.5	68	0.2534	0.3666	0.1689	0.2795	0.3094
		500	67.9	82.7	51	66.9	67.8	0.2039	0.2808	0.1424	0.2383	0.2406
		1000	64.8	81	49	64.8	64.1	0.1725	0.2323	0.1249	0.2075	0.2004
	0.9	100	69.6	82.5	46.2	61.5	69.9	0.2994	0.4326	0.1829	0.2882	0.3757
		200	68.5	84.2	45.7	63.4	68	0.2547	0.3639	0.1635	0.2764	0.3081
		500	68.3	83.8	47.9	67	67.8	0.2036	0.2788	0.1388	0.2355	0.2409
		1000	68.2	80.6	52.1	68.4	67.1	0.1769	0.2362	0.1265	0.2197	0.2047
-0.5	0.1	100	66.8	80.6	47.5	61.8	64.5	0.2984	0.4292	0.196	0.3076	0.3666
		200	68.8	81.7	49	67.7	66.8	0.2534	0.363	0.1721	0.2782	0.3037
		500	66.4	80.5	49.1	65.8	66.4	0.202	0.2767	0.1437	0.2362	0.242
		1000	65.9	82	51.6	68.4	68	0.1737	0.2356	0.1279	0.2138	0.2087
	0.5	100	69.4	83.3	47.3	63.5	68.5	0.2967	0.4284	0.1926	0.3015	0.3733
		200	68.6	83.3	47.7	65.3	69.1	0.2534	0.3642	0.1704	0.2769	0.3097
		500	69.9	84.4	52.6	69.3	69.1	0.2045	0.2814	0.1444	0.2389	0.2402
		1000	67.7	83.2	50.5	68.7	67.8	0.1754	0.2369	0.1284	0.218	0.2071
	0.9	100	65.8	78.6	43	59.4	66.4	0.294	0.4158	0.1881	0.2899	0.3641
		200	67.6	81.3	46.4	62.8	66.5	0.2531	0.3624	0.1676	0.2767	0.3036
		500	67	82.2	48.8	66.5	66.7	0.2024	0.2768	0.1414	0.2346	0.2403
		1000	64.9	82.6	50.4	66.7	66.2	0.1738	0.2356	0.126	0.2121	0.2054
0	0.1	100	69.3	81.9	46.9	62.9	68.2	0.2988	0.4302	0.1962	0.3118	0.3742
		200	66.1	81.6	48.8	62.2	67	0.253	0.3605	0.1719	0.2808	0.308
		500	63.9	79.3	48.7	64.3	64	0.1994	0.2735	0.1415	0.2293	0.2364
		1000	65.4	79.4	50.2	67.6	65.4	0.1731	0.2338	0.1277	0.2153	0.205
	0.5	100	69	79.8	45.7	61.6	66.6	0.2958	0.4244	0.1939	0.3096	0.3712
		200	66.2	82.6	45.4	63.9	66.7	0.2495	0.358	0.1689	0.2726	0.3052
		500	67.9	82.3	53.1	69.6	67.6	0.2067	0.2859	0.1476	0.2467	0.2447
		1000	66.4	81.7	51.5	69.9	67.6	0.1744	0.2367	0.1283	0.2153	0.2053
	0.9	100	71	82	50.8	64.9	70.7	0.2995	0.4304	0.1966	0.3147	0.3746
		200	68.6	82.2	50.6	66.1	67.7	0.2564	0.3714	0.1747	0.2919	0.3121
		500	65.8	82.3	51	66.8	66.8	0.2027	0.2793	0.1443	0.2375	0.2404
		1000	67.6	82	52.4	68.7	67.1	0.1742	0.2367	0.1283	0.2148	0.204
0.5	0.1	100	69.8	80.5	48.5	64.9	68.2	0.299	0.4309	0.1946	0.3111	0.3762
		200	68.7	82.1	48.7	64.9	67.8	0.2509	0.3582	0.1692	0.2747	0.3083
		500	67.7	80.6	49.1	66.7	65.1	0.2046	0.2795	0.1449	0.2381	0.2368
		1000	63.3	77.9	50.1	65.2	62.2	0.1698	0.2289	0.1238	0.2005	0.1964
	0.5	100	67.8	79.8	44.2	60	67.7	0.2947	0.4217	0.1859	0.3028	0.3682
		200	69.9	82.1	49.7	68.2	69.3	0.2561	0.3689	0.1695	0.2859	0.3117
		500	69.1	81.5	50.1	66.1	68.1	0.2035	0.281	0.1421	0.2413	0.2415
		1000	65.4	79.5	51.1	65.1	66.1	0.1712	0.2297	0.1237	0.2018	0.2013
	0.9	100	66.9	79	41.9	60.1	67	0.2949	0.4201	0.1808	0.2962	0.363
		200	67.7	81.6	46.6	64.3	66.5	0.2511	0.3622	0.1631	0.2798	0.3024
		500	67.4	81.8	50	67.8	67.5	0.2037	0.2819	0.1404	0.238	0.2424
		1000	65.8	81.5	49.5	67.8	68.3	0.1753	0.2357	0.1265	0.2191	0.2081
0.99	0.1	100	69	81.1	46.3	63.2	70.1	0.2963	0.4251	0.1885	0.3015	0.3704
		200	67.9	82	45.5	63.1	66.5	0.2536	0.365	0.1684	0.283	0.3084
		500	67.6	82.7	49.8	66.7	67.1	0.2052	0.2825	0.1439	0.2464	0.2443
		1000	65.7	80.6	48.7	66.9	64.7	0.1713	0.2313	0.1238	0.2064	0.2016
	0.5	100	67.9	80	38.5	59.1	66	0.2954	0.4239	0.1637	0.2801	0.3639
		200	67.8	82.1	45.1	62.3	66.8	0.2533	0.3624	0.1514	0.2707	0.3071
		500	66.5	83	49.7	67	67.1	0.2062	0.2864	0.135	0.2453	0.2428
		1000	66.6	79.7	44.8	66.1	64.5	0.1714	0.2303	0.1179	0.2074	0.2012
	0.9	100	69.9	81.6	33.7	59.4	68.3	0.2969	0.4254	0.1384	0.2879	0.3692
		200	67.4	81.8	38.9	60.6	65.5	0.2508	0.3605	0.1325	0.2721	0.3026
		500	69.4	84.1	46.8	66.8	70.6	0.2043	0.2814	0.1246	0.2465	0.2452
		1000	65.1	80.4	45.8	65.5	66.6	0.1728	0.2342	0.1138	0.218	0.2055

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the Gini index differences by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-d for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 26. DGP I-d: Coverage and width of 95% confidence intervals for the difference in Lorenz curve at percentile 0.5

			Coverage					Width				
ρ	λ	n	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t
-0.99	0.1	100	96.6	98.4	78.3	81.1	96.2	0.1459	0.1738	0.09	0.1099	0.1681
		200	95.3	96.4	73.6	77.9	89.7	0.1099	0.1274	0.0675	0.0816	0.1168
		500	91.1	94	67.2	71.8	89.5	0.0777	0.0899	0.0482	0.0599	0.0822
		1000	89.4	92	63.7	70.5	82.8	0.0606	0.0707	0.0382	0.049	0.0614
	0.5	100	98	98.7	71.8	76.5	96.2	0.1463	0.1741	0.0807	0.1013	0.1678
		200	96.6	97.2	70.2	75.3	91.1	0.1097	0.1275	0.0612	0.0771	0.1157
		500	92.7	94.7	65.4	72.4	89.1	0.0777	0.0901	0.0445	0.0577	0.0816
		1000	89.3	92.6	59.7	67	81.1	0.0607	0.0712	0.0359	0.0482	0.062
	0.9	100	98.8	99	67.3	75.1	97.6	0.147	0.1754	0.0699	0.0931	0.1697
		200	97.3	98.2	62.9	70.3	89.9	0.1106	0.1282	0.0549	0.0731	0.1163
		500	94.6	95.9	58.8	68.4	91.6	0.0776	0.0898	0.041	0.0558	0.0818
		1000	90.4	92.9	59.3	69.2	83.4	0.0616	0.0724	0.0345	0.0498	0.0629
-0.5	0.1	100	95.8	97.2	78.8	81.5	95.4	0.147	0.1747	0.0919	0.1109	0.1667
		200	95.3	96.6	74.8	78.7	90.1	0.1103	0.1279	0.0689	0.0825	0.1149
		500	91.1	93.4	68.3	73	89	0.0773	0.0894	0.0487	0.0602	0.082
		1000	89	92.1	65	71.7	85.9	0.0608	0.072	0.039	0.0503	0.0643
	0.5	100	97.5	98	78.6	81.3	95.8	0.1462	0.1743	0.0891	0.108	0.1684
		200	95.5	97.1	76.8	78.8	92.6	0.11	0.1276	0.0669	0.0804	0.1182
		500	94	96.2	71.3	76	92	0.0777	0.0903	0.0478	0.0594	0.0811
		1000	90.3	93.1	65.1	72.5	84.2	0.0612	0.072	0.0385	0.0506	0.0635
	0.9	100	96.8	97.8	75.3	80.1	96.8	0.1458	0.173	0.0862	0.1048	0.1673
		200	95.4	96.5	71.5	75.6	90.2	0.1105	0.1278	0.0652	0.079	0.1158
		500	91.9	94.2	66.6	72.6	89.7	0.0771	0.0893	0.0463	0.0576	0.0819
		1000	88.6	92.9	62.1	68.8	82.9	0.0609	0.0716	0.0375	0.0489	0.0627
0	0.1	100	97	97.8	81.1	84	96	0.1469	0.175	0.0923	0.1123	0.1696
		200	94.8	95.3	75.4	78.2	89.7	0.1106	0.1281	0.0694	0.0835	0.1179
		500	91.2	93.5	67.7	72.8	88.4	0.0767	0.0886	0.0485	0.0593	0.081
		1000	88.5	91	64.1	70.9	82.7	0.0607	0.0714	0.0392	0.0511	0.0634
	0.5	100	96.2	96.8	79.4	83	94.6	0.1458	0.1735	0.0918	0.1122	0.1675
		200	95.4	96.5	76	79.2	90.8	0.1093	0.1266	0.0685	0.0819	0.1164
		500	91.6	93.9	68.6	74	88.2	0.0781	0.0911	0.0495	0.0614	0.0823
		1000	89.4	92.3	65.7	72.1	83.6	0.0609	0.0716	0.0392	0.0505	0.0626
	0.9	100	97.7	98.3	81.6	84.9	97.2	0.1468	0.1747	0.0922	0.1126	0.1695
		200	94.8	96.2	73.8	77.7	90.5	0.1106	0.1286	0.0694	0.0843	0.1181
		500	89.9	92.8	67.2	71	87.8	0.0774	0.0895	0.049	0.0602	0.0813
		1000	89.4	91.9	64.2	71.5	84.2	0.061	0.072	0.0392	0.0508	0.0626
0.5	0.1	100	96.1	97.6	79.9	82.8	96.4	0.1467	0.1744	0.0913	0.1109	0.1708
		200	94.9	96.1	73.5	76.6	88.8	0.1097	0.127	0.0682	0.0816	0.1179
		500	91	93.7	67.6	72.8	87.6	0.0781	0.0906	0.049	0.0602	0.0809
		1000	87.7	90.4	64.2	69.8	81.4	0.0601	0.0704	0.0382	0.0484	0.062
	0.5	100	97.1	98.3	77.1	80.4	96.3	0.1458	0.1733	0.0875	0.1075	0.1688
		200	96.1	96.6	73.8	78	89.4	0.1108	0.1286	0.0666	0.081	0.1179
		500	92.8	94.1	68	73.7	90.4	0.0776	0.09	0.0473	0.0593	0.0814
		1000	88.6	92	63.2	69.3	84.2	0.0604	0.0708	0.0376	0.0481	0.063
	0.9	100	96.9	97.8	74.7	79.6	95.9	0.1463	0.1735	0.0846	0.104	0.1671
		200	95.6	96.4	71.8	76.3	90.2	0.1095	0.1268	0.0639	0.079	0.1154
		500	92.4	94.4	66.6	71.8	89.7	0.0777	0.0901	0.0461	0.0579	0.0829
		1000	89.6	93.6	62.2	69.3	84.6	0.0613	0.0719	0.0376	0.0503	0.0635
0.99	0.1	100	96.7	97.5	78.8	83.6	95.7	0.1463	0.1738	0.0881	0.1075	0.1683
		200	95.5	96.7	72.1	75.4	91.3	0.1104	0.1278	0.0664	0.0811	0.1181
		500	92.7	94.9	67.9	73	89.3	0.0778	0.0902	0.0475	0.0605	0.0818
		1000	88.4	92	62	69.7	82.3	0.0604	0.0708	0.0375	0.0486	0.0618
	0.5	100	98.6	98.9	67.2	73.6	96.6	0.1464	0.174	0.0723	0.0923	0.1658
		200	97.2	97.8	63.6	69.8	90.9	0.1102	0.1277	0.0555	0.0711	0.1168
		500	94.7	95.3	60.5	68	89.6	0.0781	0.0911	0.0413	0.0557	0.0815
		1000	90.8	93.8	54.4	65.4	82.5	0.0604	0.0705	0.0334	0.0467	0.0619
	0.9	100	99.8	99.9	52.7	66.3	97.8	0.1463	0.1737	0.0535	0.0804	0.1668
		200	98.5	98.4	51	62.5	91.5	0.1097	0.1273	0.0434	0.0647	0.1154
		500	96.6	97.1	54.2	65	93.1	0.0776	0.0901	0.035	0.0548	0.0823
		1000	90.8	93.6	50.4	62.8	84.1	0.0606	0.0713	0.0299	0.0485	0.0623

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the difference in Lorenz curve at percentile 0.5 by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot) and robust-t method with four groups (t). The data are generated according to DGP I-d for given overlap portion λ and sample size n and ρ in Gaussian copula (5.3). The numbers are based on 1000 replications with 999 bootstrap repetitions each.

B.4. Additional simulation results beyond overlapping samples

DGP II-A': (Violate (A2)) $\{X_{k,i}\}_{i=1}^{n} = \{U_{k(i)}\}_{i=1}^{n}$ for $k = 1,2$, where $U_{k(i)}$ is the i-th order statistic. $\{U_{1,i}\}_{i=1}^{n} \overset{iid}{\sim} \mathrm{SM}(1, 1.6971, 8.3679)$, $\{U_{2,i}\}_{i=1}^{n} \overset{iid}{\sim} \mathrm{SM}(0.4, 1.4, 1.5)$, $\rho = 0.95$ or $\rho \sim U(-1,1)$. The sample sizes are $n = 100, 200, 500, 1000$. The overlap portions are $\lambda = 0.5, 1$.

Simulation results for additional DGPs are presented below. The main findings, however, remain unaffected.

DGP II-a: (Violate (A2)) $\{X_{k,i}\}_{i=1}^{n} = \{U_{k(i)}\}_{i=1}^{n}$ for $k = 1,2$, where $U_{k(i)}$ is the i-th order statistic. $\{U_{1,i}\}_{i=1}^{n} \overset{iid}{\sim} \mathrm{SM}(1, 1.6971, 8.3679)$, $\{U_{2,i}\}_{i=1}^{n} \overset{iid}{\sim} \mathrm{SM}(0.4, 2.8, 1.7)$, $\rho = \mp 0.95$ or $\rho \sim U(-1,1)$. The sample sizes are $n = 100, 200, 500, 1000$. The overlap portions are $\lambda = 0.5, 1$.

DGP II-b: (Violate (A3)) $\{X_{2,i}\}_{i=1}^{2n} \overset{iid}{\sim} \mathrm{SM}(0.4, 2.8, 1.7)$ and $\{X_{1,i}\}_{i=1}^{n}$ where $X_{1,i} = X_{2,2i-1} - X_{2,2i}$. $n = 100, 200, 500, 1000$.

Table 27. DGP II-A': coverage and width of 95% confidence intervals for mean differences for non-overlapping samples

ρ	λ	n	Coverage						Width					
			IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	t-Shuffle	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	t-Shuffle
0.95	0.5	100	97.9	98.9	85.2	90.1	100	94.7	0.4335	0.6886	0.2874	0.5209	1.0759	0.4911
		200	97.4	98.6	87.1	90.9	100	95.3	0.3215	0.4412	0.2158	0.327	1.0354	0.3681
		500	97.6	98.7	87.9	91	100	95	0.2132	0.2697	0.1454	0.2015	1.0008	0.247
		1000	98.6	99	90.5	92.2	100	96.2	0.156	0.1882	0.1076	0.1396	0.9889	0.1778
	1	100	98.4	99.4	82.9	90.6	100	96.2	0.4138	0.5906	0.2338	0.461	1.8181	0.4738
		200	98.7	99.3	84.8	90.2	100	96.8	0.3155	0.4495	0.1868	0.3676	1.8272	0.3625
		500	99	99.2	88.1	91.8	100	96.8	0.2199	0.3121	0.1369	0.2546	1.8487	0.2527
		1000	98.7	99.1	88.4	92.1	100	97.6	0.162	0.221	0.1033	0.1761	1.8542	0.1897
U(-1,1)	0.5	100	94.5	96.5	80.9	85	100	92.9	0.425	0.6418	0.2811	0.4821	1.0635	0.4845
		200	97.5	98.4	86.8	90.3	100	94.2	0.3138	0.4136	0.21	0.3055	1.0326	0.3518
		500	97.2	98.3	87.1	89	100	94	0.2195	0.3167	0.1513	0.2509	1.0099	0.2543
		1000	97.2	98.1	89.2	91.9	100	95	0.1597	0.1999	0.1108	0.1509	0.9886	0.1828
	1	100	95.9	97.7	75.7	82.2	100	93.4	0.4183	0.6036	0.2385	0.4905	1.8063	0.4714
		200	97.8	99	81.5	85.6	100	95.9	0.3267	0.5085	0.1969	0.4362	1.8687	0.3799
		500	97.2	98.6	82.3	85.8	100	95.2	0.2119	0.2753	0.1301	0.2126	1.8329	0.2422
		1000	97.9	98.5	85.3	87.2	100	94.4	0.1608	0.2082	0.1022	0.1621	1.8495	0.1861

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean difference by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot), robust-t method with four groups (t) and robust-t method using the random split with four groups (t-Shuffle). The data are generated according to DGP II-a with overlap portion λ (if applicable) and sample size n. The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 28. DGP II-a: coverage and width of 95% confidence intervals for mean differences for non-overlapping samples.

ρ	λ	n	Coverage						Width					
			IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	t-Shuffle	IM-Asym	IM-Boot	OS-Asym	OS-Boot	t	t-Shuffle
-0.95	0.5	100	99.2	99.5	76.9	74.9	100	96.8	0.1662	0.173	0.0752	0.0759	0.5143	0.1606
		200	99.1	99.4	77.8	77	100	96	0.1186	0.1211	0.0532	0.0537	0.5088	0.1136
		500	99.6	99.2	76.2	76.2	100	95.9	0.0754	0.0762	0.0336	0.0338	0.507	0.072
		1000	99.4	99	77.9	77	100	96.9	0.0533	0.0535	0.0237	0.0238	0.5045	0.0516
	1	100	98.3	98.8	21.7	24	100	95.1	0.1672	0.1742	0.0209	0.0294	0.8485	0.1635
		200	98.5	98.8	16.9	18.7	100	94.2	0.1182	0.1207	0.0124	0.0164	0.8481	0.1139
		500	98.2	98.4	15.9	19.4	100	94.8	0.0754	0.076	0.0071	0.0094	0.8508	0.0735
		1000	97.9	97.9	13.4	17.7	100	94.5	0.0534	0.0536	0.0048	0.006	0.8517	0.0512
0.95	0.5	100	100	100	93.4	92.8	100	99.5	0.167	0.1738	0.0745	0.0749	0.5172	0.1612
		200	100	100	94.2	94.3	100	99.7	0.1183	0.1209	0.0526	0.0527	0.508	0.1138
		500	100	100	94.9	95	100	99.3	0.0753	0.0759	0.0333	0.0336	0.5052	0.0727
		1000	100	100	95.4	95.3	100	99.4	0.0535	0.0538	0.0237	0.0238	0.5053	0.0512
	1	100	100	100	62.8	61.5	100	100	0.1672	0.1746	0.0149	0.0218	0.846	0.1616
		200	100	100	65.8	65.2	100	100	0.1186	0.1211	0.0099	0.0143	0.848	0.1166
		500	100	100	63.9	65.6	100	100	0.0755	0.0762	0.0064	0.0092	0.8511	0.0726
		1000	100	100	67.9	68.9	100	100	0.0535	0.0538	0.0045	0.0058	0.852	0.0521
U(-1, 1)	0.5	100	99.9	99.8	85.9	84.3	100	97.9	0.1659	0.1728	0.0749	0.0758	0.5154	0.1634
		200	99.7	99.6	83.3	83.2	100	98.1	0.1192	0.1218	0.0536	0.0538	0.5132	0.1155
		500	99.9	99.9	84.4	84.4	100	98.9	0.0757	0.0764	0.0338	0.034	0.5087	0.0741
		1000	99.7	99.6	83.3	82.5	100	98.1	0.0534	0.0536	0.0237	0.0238	0.5051	0.0517
	1	100	99.8	99.8	28	28.8	100	98	0.1671	0.1739	0.0207	0.0281	0.8465	0.1616
		200	99.7	99.6	24.5	27.4	100	97.7	0.1187	0.1212	0.0128	0.018	0.8487	0.1161
		500	99.6	99.6	22.9	26	100	98.2	0.0755	0.0762	0.0072	0.0095	0.8516	0.0728
		1000	99.8	99.8	22.6	24.5	100	97.8	0.0534	0.0536	0.0048	0.006	0.8512	0.051

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean difference by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot), robust-t method with four groups (t) and robust-t method using the random split with four groups (t-Shuffle). The data are generated according to DGP II-a with overlap portion λ (if applicable) and sample size n. The numbers are based on 1000 replications with 999 bootstrap repetitions each.

Table 29. DGP II-b: coverage and width of 95% confidence intervals for mean differences for non-overlapping samples.

n	Coverage						Width					
	IM-Asym	IM-Boot	OS-Asym	OS-Boot	*t*	t-Shuffle	IM-Asym	IM-Boot	OS-Asym	OS-Boot	*t*	t-Shuffle
100	99.8	99.8	86.5	76.5	97.2	97	0.1873	0.191	0.0946	0.097	0.187	0.1887
200	99.9	99.9	88.2	79.2	97.3	98.1	0.1334	0.1346	0.0674	0.0683	0.1329	0.1343
500	99.9	99.8	85.3	75.9	97.5	97.5	0.0848	0.0851	0.0428	0.0433	0.0849	0.0842
1000	99.8	99.6	86.5	77.8	97.9	97.8	0.0596	0.0596	0.0301	0.0303	0.0595	0.0608

Note – The table provides coverage rates and widths of confidence intervals of level 95% for the mean difference by asymptotic intersection method (IM-Asym), bootstrap intersection method (IM-Boot), asymptotic method with overlapping samples (OS-Asym), bootstrap method with overlapping samples (OS-Boot), robust-*t* method with four groups (t) and robust-*t* method using the random split with four groups (t-Shuffle). The data are generated according to DGP II-b with sample size n. The numbers are based on 1000 replications with 999 bootstrap repetitions each.