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Harvest Age and Species Choice in Forestry

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Asset Retirement with Infinitely Repeated Alternative Replacements:
Harvest Age and Species Choice in Forestry[*]

Skander Ben Abdallah[†], *Pierre Lasserre*[‡]

Abstract

At what age should productive assets be retired? How should replacements be chosen when they differ in their uncertain ability to generate future incomes? As a particular version of that problem, we study the tree harvesting decision with two possible replacement species whose values as timber are stochastic and whose growth functions are deterministic. In the single-rotation (Wicksell) problem starting with a bare piece of land (an empty shop), it is optimal to choose and plant one species immediately if its current value is sufficiently high relative to that of the other species (the alternative equipment). However, if the species are insufficiently price-differentiated, it is preferable to leave the land vacant (the shop empty) despite the opportunity cost of doing so. In the repeated version of the problem, it is never optimal to leave the land bare provided the cost of replacement is null. Furthermore, the optimal harvest (tree retirement) age not only depends on the price and current productivity of the trees in place but also on the price and productivity of the other species, because it may replace the current one. The harvest age reaches a peak at some critical threshold of the relative price that signals the necessity to switch to the alternative species; indeed this is when the opportunity cost of choosing one alternative replacement over the other is the highest. The land value (and also the value of the firm) is similar to an American option with free boundary, infinite expiry period, and endogenous payoff. The paper highlights the opportunity cost of alternative replacement options, and the central role of their volatility in both asset-retirement and replacement-choice decisions. All results are derived analytically; a numerical treatment by the penalty method completes the resolution.

Keywords: Real options; stochastic prices; American option; asset retirement; replacement option; penalty method; forestry; Faustmann; alternative species; rotation.

Codes JEL/JEL Codes: C61; D81; G11; G13; Q23.

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1. Introduction

At what age should current productive assets be retired when alternative replacements exist that differ in their uncertain ability to generate future incomes? To focus on the opportunity cost of choosing between alternatives, this paper assumes the direct cost of replacement to be zero so that either choice of replacement is economically desirable in the absence of alternative. However choosing one option over the other may prove to be a mistake *ex post* if the latter turns out to be more profitable. Retirement and adoption decisions must keep the risk of making such mistakes reasonably low. Applying the proper decision rule for current and future decisions will maximize firm value or rent.

Our paper addresses these questions by focusing on extensions of the conventional forestry economics model. This archetypal investment problem has played a central role in the development of economic investment theory and is associated with famous figures such as Jevons, Wicksell or Von Thunen. As Samuelson (1976) pointed out not without cruelty, early and more recent work on that model provided the economic profession with ample opportunities to learn from the mistakes of such brilliant economists as Irving Fisher, Harold Hotelling, or Kenneth Boulding while pondering on the contribution of the accounting profession to economics (see Amacher, 2015). Economists had erred by failing to recognize that occupying land imparts an opportunity cost to delaying harvest.

In the infinite horizon version of our extended model, trees are the equipment of the forest firm, whose capital also includes the forest land, of endogenous value. Existing trees are 'retired' by harvesting at an optimally chosen age and two alternative species may be planted to replace them. These replacements may differ in their ability to generate future income and their current price conveys information on the value of future harvests. Leaving the land bare for any amount of time and mixing species for diversification are possible. This entails more sophisticated harvesting (or retirement) and replacement decisions than had been considered before. In particular the importance as opportunity cost of an alternative replacement option is clarified and the central role of its volatility in both asset-retirement and replacement-adoption decisions is brought to light.

Forest management involves time, uncertainty, and irreversible decisions with consequences in the future. It also exemplifies investments that open up new options: harvesting a tree opens the option of planting a new one or using the land differently. Faustmann (1849) gave forestry economics its foundations by addressing the question: at what age should a stand of even-aged trees be harvested? He did so under the assumption of a known constant timber price by comparing the net marginal benefits from letting timber grow further, to the opportunity cost of existing trees plus the opportunity cost of occupying the land, itself a function of timber management decisions.

Faustmann's original problem has been refined and generalized in many ways. In this paper we focus on the availability of alternative species or land uses to replace the trees being harvested. We show how this choice should be made and timed, how it affects the harvesting decision, and how it ultimately determines the value of the firm.

While species choice or land use is clearly important for the forester, that question has much wider relevance and may be viewed as illustrating the resolution of a general investment problem. The question of choosing a harvest age is akin to that of deciding at what age an equipment should be retired; the declining rate of growth of trees is akin to the declining productivity of equipment with usage or obsolescence. The species choice is similar to the choice of alternative technologies or alternative activities for replacing the retired equipment, where the timber price may also represent the ability of alternatives to generate income. Thus we introduce alternative assets in situations where decisions open up new options in a process that repeats itself indefinitely, and must be reevaluated at each instant. The options to be considered remain available forever and include reinstalling a new version of the asset currently in place or adopt the alternative one. This most common practical situation has not been fully investigated theoretically before, although the real option literature, the forestry, and the asset replacement literature have gone some way toward solving related problems.[2]As will become apparent from a

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[2] Although the possibility of abandoning timber production for some alternative forest use or activity such as agriculture or housing is frequently envisaged, we are aware only of cases where the switch is considered irreversible.

brief review of the relevant literature and from the description of our results, we combine and complement these strands of work in a simplified model that brings out clear and intuitive results and decision rules.

The real option approach has been widely applied in natural resource exploitation and management. In the real option framework, a typical investment involves an optimum stopping rule that defines the date at which conditions have become favorable enough to justify committing resources toward a project irreversibly. A typical result is that more uncertainty postpones investment although it increases asset value.

Applying this approach, a number of studies (e.g. Brazee and Mendelsohn, 1988; Clarke and Reed, 1989; Reed and Clarke, 1990; Thomson, 1992) examine the optimal harvesting age in forestry under stochastic timber price.[3] These studies show that when timber price or stand value follow a Brownian motion and regeneration costs are absent, the optimal harvest age is insensitive to price despite the fact that the price is not assumed constant as in the deterministic case. However, when regeneration costs are present, the choice of the harvest age can be improved by exploiting the stochasticity of the price, which provides an opportunity to delay regeneration costs until future prices are observed. Platinga (1998) shows that the option value to delay harvest when timber prices are stationary stems both from the level of the current stumpage price relative to its long-term mean and from the stand value relative to the fixed regeneration cost. Over time, applications have been extended to include more and more features, such as differentiated timber prices (Forboseh et al., 1996), a variety of price processes (Alvarez and Koskela, 2007), uneven-aged management (Haight, 1990), multi-species stands under changing growth conditions (Jacobsen and Thorsen, 2003), stochastic discount rate (Alvarez and Koskela, 2005), endogenous forest area (Sahashi, 2002), the value of carbon storage (Ekholm, 2015), and many others referred to in Amacher et al. (2009).

The real option literature has treated situations where a one-shot action simultaneously involves the timing of an investment and choosing between alternative opportunities. These alternative opportunities may differ with respect to investment costs and

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[3] Willassen (1998) dismisses the optimal stopping methodology and uses impulse control.

output as in Geltner et al. (1996) and Decamps et al. (2006) or with respect to benefit trajectories as in Kassar and Lasserre (2004). An interesting result in such situations is that a new reason for postponing action arises. When the alternatives are too close to each other and uncertain, the decision maker may choose to wait in order to avoid choosing an alternative that might prove to be less desirable than another candidate in the future. This inaction may be optimal although each asset, taken in isolation, would satisfy the requirements for immediate investment under conditions of irreversibility and uncertainty.[4]

In this paper, we consider alternative options that are renewed and repeated indefinitely. In the asset replacement literature, Richardson et al. (2013) model replacement as infinitely repeated options complicated by lead time uncertainty, but not involving any choice between alternatives. Ye (1990) and Mauer and Ott (1995) analyze infinitely repeated replacement options for alternative equipments whose productivity is constant over time but whose maintenance and operation cost follows a Brownian motion. The cost is reset to a known initial value when the alternative asset is installed, but is not subject to any change related to its use by the firm. As a result the value of the option depends on that single variable.

For Malchow-Møller and Thorsen (2005; 2006), whose paper is perhaps the closest to our's, it is the productivity of installed technology relative to newer available technology which is at the core of modern replacement decisions. Malchow-Møller and Thorsen (2005, page 1028) analyze a model with two alternative technologies the productivities of which follow geometric Brownian motions. The installed technology can be replaced

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[4] Although the forestry literature has considered choices between alternative options such as the arbitrage between investing in a stand and preserving old growth (Reed, 1993; Conrad, 1997) or the watershed (Abildtrup and Strange, 2000), no attention has been devoted to the hysteresis possibly induced in such circumstances. Furthermore, when simultaneous alternatives were considered, the future stand value was treated as exogenous, independent of the current choice. For instance, Thorsen (1999) analyses the choice of tree species for afforestation as a real option problem, and Thorsen and Malchow-Møller (2003) extend it to a two-option problem with two mutually exclusive options (two tree species), where exercising one option implies losing the other. Jacobsen (2007) goes one step further: upon harvest, the current stand (of spruce) may be allowed to regenerate naturally and costlessly, or may be replaced with oaks once and for all. However, it is assumed that at some finite time oaks will be definitely planted so that the problem eventually simplifies to the single-species problem of Faustman with stochastic price.

repeatedly by the alternative one at a certain fixed cost. Since both technologies evolve in an exogenous stochastic way, either may become temporarily more productive in a never ending race against obsolescence. However, while the installed technology may become obsolete, this would be the only reason for replacement. Equipment does not physically depreciate. This is a major difference with our model where trees grow old and need to be replaced even if they belong to the species that appear the most promising at harvest time. Technically this difference implies that, while their model is time invariant, our's depends on both time and the stochastic state variable.

To our knowledge our paper is the first one solving analytically an infinitely repeated choice between replacement options that depend on two state variables, namely (after adequate transformation of the model) the age of the trees in place and the relative value of the species as timber. Given an existing stand of even-aged trees, an expected value maximizer chooses the harvest age. Then, she decides which species or which proportion of species should be used to repopulate the land, where timber prices evolve stochastically, possibly with some correlation. Then again, and forever, harvests times and species must be chosen optimally. To draw again on the analogy with equipment retirement and replacement, equipment may be retired at any age and replaced with any of two alternative technologies where the efficiency or price of each technology evolves stochastically as suppliers improve their products or demand for their characteristics evolves. It is not certain that one type will dominate the other for long, let alone forever, but sooner or later the new equipment will need to be replaced again because it ages or becomes obsolete. The problem of retirement and replacement repeats itself infinitely and its resolution determines the value of the firm endogenously.

The setting and assumptions of the forest model are introduced in Section 2. In Section 3, we investigate the case of a single rotation, also known as Wicksell's tree harvesting problem already analyzed for a single species of stochastic price by Willassen (1998). With two species, one should consider the species choice and the timing of that decision, before considering harvest. A form of hysteresis arises: under some conditions about relative species prices, the decision maker waits rather than establishing a crop, despite the fact that each species would be worth planting in isolation.

In Section 4 we extend the analysis to multiple rotations. The decision maker must decide at what age the trees of the current stand must be harvested; after harvest she must decide whether she should establish a new crop or wait; when she makes such decision, she must choose one, or the other, species. This process is repeated infinitely. We find that the hysteresis of the Wicksellian problem disappears. This extreme contrast with the one-shot case is due to the absence of management cost. As will be seen, it is part of a wider phenomenon identified by Malchow-Møller and Thorsen (2005) according to which introducing repeated options reduces the value of waiting as it dilutes the irreversibility effect.

The qualitative properties of the decision rules and value functions are described analytically and presented in a number of propositions.[5] The stand value is similar to the value of an American option with a free boundary, an infinite expiry date, and an endogenous payoff. The optimum harvest age increases when the relative price approaches some threshold value signalling the necessity to switch to the alternative species. This is because the decision maker would rather wait than plant the wrong species; letting existing trees grow older is a way to postpone the choice, allowing the prices to become more informative about the species to be chosen. Conversely, if the relative price exceeds the switching threshold, the risk of making a mistake by switching diminishes as the relative price moves further away from the threshold, so that the reason to postpone harvest becomes less compelling and the optimum harvest age diminishes. This possibility to postpone harvest explains why the extreme hysteresis of the one shot problem is replaced, when harvests can be infinitely repeated, with an increase in the harvest age that has nonetheless a similar purpose: postponing the date at which new trees are established in the hope to allow prices to become more informative.

Whether as contributions to forest economics or, more generally, as contributions to

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[5] The assumption that prices follow geometric Brownian motions helps obtain some of the analytical results. However the numerical treatment does not depend on that assumption and the qualitative properties of the solution, as well as the intuitive understanding arising from the analytical results are robust to changes in it.

asset retirement analysis or investment theory, these results are new in highlighting the role of the uncertainty surrounding opportunity costs in the determination of threshold values for investment and timing. They also formalize the complex link between asset retirement and replacement.

Before concluding in Section 5, we complete the analysis of Section 4 with some comments on the effects of volatility and uncertainty on the harvest decision and the value of the forest. These remarks are based on a numerical resolution applying the penalty method (Zvan et al. 1998) simultaneously to the stand expectation value function of each species, while determining the land expectation value by a Newton iterative process. This allows explicit comparisons with the non stochastic solution of the model obtained analytically and presented in Ben Abdallah and Lasserre (2015).

2. GENERAL SETTING AND ASSUMPTIONS

We study the decision by an expected value maximizer to establish one, or any mix, of two alternative tree species or value generating activities P and P' on a plot of bare land. To simplify the exposition, we are going to refer to alternative tree species in the sequel, although adjustments allowing other interpretations are possible. The timber price of species P (respectively P') follows a geometric Brownian motion (GBM) with constant drift μ (resp. μ') and constant volatility σ (resp. σ'):

$$dp = \mu p dt + \sigma p dz, \tag{1a}$$

$$dp' = \mu' p' dt + \sigma' p' dz. \tag{1b}$$

where time indices have been omitted, $dz = \varepsilon\sqrt{dt}$ and $dz' = \varepsilon'\sqrt{dt}$ are the increments of Wiener processes, and ε and ε' are standardized Gaussian white noises whose correlation is ρ.

Timber prices rising at a strictly positive rate were analyzed by Lyon (1981) who justified them on the ground that there is a mining dimension to forest exploitation, but that rising scarcity is moderated by the renewability of the resource. Some historical data (Newman et al, 1985) weakly support this strong version of the constant rate assumption although evidence based on over 100 years of stumpage data from Sweden (Hultkrantz et al., 2014) rather points to a mean reverting process.[6] As these authors remark, this is important for decision making and may imply that forest assets are different from financial assets in that timber should be sold when prices are above their mean if mean there is. However, for the current research, what matters is that one species may become more attractive to plant and harvest than the other one and vice versa, repeatedly. With prices driven by GBM's this depends on the evolution of the relative price.[7] Given the weakness of the evidence on timber price levels, it is very unlikely that the random walk assumption could be rejected for relative species prices. Indeed the prices of various timber species relative to other species may evolve differently because of differences in demand or, e.g., of their different abilities to sequester carbon (Sohngen and Mendelson, 1998), allowing the commercial values of alternative species to overtake each other, perhaps forever, but also possibly for finite and uncertain periods. This last aspect is at the root of our investigation and the GBM with correlation is a beautiful laboratory for its analysis.

In the rest of the paper variables that depend on time are indexed unless they are considered at the current date defined as $t = 0$, in which case the time index is omitted. While $\delta = r - \mu > 0$ and $\delta' = r - \mu' > 0$[8] are constant parameters, where r is the discount rate, the relative price $\theta = \frac{p'}{p}$ is time variable[9]. Thus the price of one species may be lower or higher than the price of the other species at anyone time and that situation may reverse itself any number of times. When $\mu = \mu'$ reversals are purely random events; when one of the drifts is higher than the other one the probability that the high-drift species price-dominates the other one tends to increase over time, although the

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[6] For an interesting discussion of both the evidence and its implications, see Hultkranz et al. (2014).

[7] With mean reverting prices similar situations will arise if the means to which prices are attracted evolve over time, so that relative means change. The mathematical analysis would change but the basic idea would be similar.

[8] We assume that $\delta > 0$ and $\delta' > 0$ which guarantees that it will not be optimal to delay the harvest decision forever provided the growth in commercial volume vanishes or become negative beyond some tree age.

[9] The relative price θ is a GBM as $\frac{d\theta}{\theta} = \left(\mu' - \mu + \sigma^2 - \rho\sigma\sigma'\right) dt + \sigma' dz' - \sigma dz$ with drift $\mu_\theta = \mu' - \mu + \sigma^2 - \rho\sigma\sigma'$ and volatility $\sigma_\theta^2 \equiv \sigma^2 - 2\rho\sigma\sigma' + \sigma'^2$.

possibility of a reversal only vanishes in the limit when the relative price tends toward infinity. If both volatilities are zero and $\mu = \mu'$, the relative price remains constant so that one species should be preferred to the other one forever. If both volatilities are zero and one drift is higher than the other one, it is certain that the high-drift species will override the other one at some known time if its harvest value is not higher already. The non stochastic problem, *i.e.* when $\sigma = \sigma' = 0$, is solved in Ben Abdallah and Lasserre (2015); therefore, we limit the present paper to the stochastic problem and assume from now on that $(\sigma, \sigma') \neq (0, 0)$. In addition, we assume that timber prices are not completely correlated *i.e.* $\rho \neq 1$; otherwise prices follow parallel trajectories, so that the higher price trivially remains so forever; it is then optimal to invest immediately in the asset with the best harvest value. This is summarized in the following assumption.

Assumption 1 *The stochastic price processes* (1*a*) *and* (1*b*) *are such that* $\rho \neq 1$, $\delta > 0$ *and* $\delta' > 0$, *and* $(\sigma, \sigma') \neq (0, 0)$.

Whether of Species P or P', trees grow according to their age a. The timber volume functions $V(a)$ and $V'(a)$ of Species P and P' may differ from each other but have in common the following properties:[10]

Assumption 2 *There exists* $\underline{a} > 0$ *and* $\underline{a}' > 0$, *such that the timber volume functions are continuous on* $[0, +\infty[$, $V(a) = 0$ *on* $[0, \underline{a}]$, $V'(a) = 0$ *on* $[0, \underline{a}']$; $V(a)$ *and* $V'(a)$ *are positive, continuous, differentiable and concave on* $[\underline{a}, +\infty[$ *and* $[\underline{a}', +\infty[$ *respectively. In addition,* $\lim_{a \to +\infty} V_a(a) = 0$ *and* $\lim_{a \to +\infty} V'_a(a) = 0$.

Empirical commercial volume functions are usually non decreasing functions of tree age (see Payandeh (1991) for examples of yield or volume tables and analytic formula). Their level is negligible at low age value, they have a convex initial part and become concave once the trees have reached some strictly positive age.[11] Assumption 2 ensures

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[10] Throughout the paper the notation " ' " applied to a function will refer to the alternative species while first or second derivatives of functions will be denoted by indices. Thus, for variables a and p, $G_a(a, p)$ denotes the partial derivative of the function $G(a, p)$ with respect to a and $V'_{aa}(a)$ denotes the second derivative of the function $V'(a)$.

[11] This implies that it is not optimum to harvest at an arbitrarily low age as would be the case in the absence of fixed cost if returns to age were decreasing at low values.

that these stylized properties are satisfied while avoiding delicate and economically trivial complications associated with the non convexity of the volume functions at low tree ages. The assumption that growth goes to zero rather than becoming negative at high ages is not realistic. However it is harmless and without impact on the results. As discussed after Proposition 3 the portion of the function that is relevant for harvest decisions is when growth is still vigorous enough to match the opportunity cost of holding standing trees rather than selling them as timber.

We assume that operational costs (while trees are growing) and harvesting costs are either nil or accounted for by measuring the growth function as net of these costs. To the extent that these costs only depend on tree age and are not stochastic or time dependent (as opposed to age dependent), this assumption is innocuous. For example, if young trees require more management interventions while older trees get vulnerable to fungi and pests, or to age dependent cracks and miss-coloration, these costs can be evaluated in terms of lost volume and integrated into the construction of volume functions. Thus functions V and V' reflect best sylvicultural practices and are similar in that respect to production functions.

A more consequential assumption is that planting and other initial investment costs are zero. It is clearly not satisfied in practice where soil preparation and plantation are often important sylvicultural interventions after harvest. It is also almost certain that additional costs are associated with species changes. As will be stressed in due time, this assumption has an effect on some of the results.[12] We make it for two reasons. The first one is theoretical. A key element of real options decisions is the irreversible commitment of resources at the time a project is undertaken. When these resources take the form of an irreversible investment cost, and the future returns of the asset are uncertain, they create a deferral option that is well understood.[13] This paper focuses on the option to choose, once or an infinite number of times, between two alternatives. This option is best analyzed in the absence of sunk costs in order to focus on its specific feature,

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[12] We thank an anonymous referee for helping us clarify the role of this and other assumptions.

[13] See the references in the introduction.

which is that exercizing it kills the opportunity to choose the alternative possibility. The second reason is that eliminating planting and replacement costs allows to go much further into the analytical resolution of the problems, thus providing a better description and understanding of the impact of the switching option on harvest and replacement decisions. Nonetheless, the numerical method developed to illustrate and quantify the solution can be adapted to a model involving a positive planting or investment cost; this numerical method is presented in Appendix B.

As pointed out by an anonymous referee, the model implies a further, hidden, assumption that matters in both its forestry and its equipment replacement interpretations. This assumption is that the volume functions V and V' (or the equipement productivities) are independent of the presence or the absence of the alternative on the forest tract or in the shop. However forest managers know that thinning actions may favor one species when several species are present simultanously and plant managers know that equipment productivity may differ when operators are exposed to different types of equipement rather than only one. Assumption 2 rules out such possibilities.

3. Wicksell's problem with two tree species

The Wicksellian tree harvest problem refers to the problem of choosing the age at which a stand of even-aged trees will be harvested when a single harvest is considered and the land is assumed to be of negligible value beyond this unique harvest. The optimal harvesting age is determined by the well-known Wicksellian rule under which the marginal value growth of the trees must be equal to the opportunity cost of holding them.

We modify the stochastic version of Wicksell's problem in two essential ways. First we start with bare land and consider the tree planting decision. Second we study a situation where two tree species, not simply one, are available. The reasons why these new features are important are the following. First, since the timber value of each species evolves stochastically, the revenues derived upon harvest depend on the species initially planted and on the price reached at harvesting. The harvest value of one species may overtake the other one, implying that the decision maker may regret the initial choice.

Second, the availability of more than one species opens up the possibility of diversification that we show is not optimal in the following proposition.

Proposition 1 *When a single crop consisting of any combination of two tree species may be planted by an expected value maximizer, it is preferable to plant either one, or the other, species rather than diversify.*

Proof. See the appendix. ■

Proposition 1 indicates that the stand value, which is the expected discounted value of the harvest given the use of an optimal rule for choosing the harvest age, is highest when one single species, rather than a combination of the two species, is established on the plot. This results relies on the fact, implicit in Assumption 2, that the growth function of a species is the same whether trees of the other species are mixed with it or not. However, the forest management literature describes how an initial mix of species, say half P, half P', can be managed over time by selective thinning in such a way that the total volume production is largely unaffected while one or the other species is progressively favored according to its commercial desirability. Such practice, ruled out by our assumptions, effectively defers species choice (see Jacobsen and Thorsen, 2003 for an example of this feature in a numerical model).[14]

Let $G(p, a)$ ($G'(p', a)$) be the stand value function when species P (resp. species P') is in place, the age of the trees is a, and the current price of the species in place is p (resp. p'):

$$G(p,a) = \max_s E\left[e^{-rs}V(a+s)p_s\right], \tag{2a}$$

$$G'(p',a) = \max_s E\left[e^{-rs}V'(a+s)p'_s\right]. \tag{2b}$$

Consider $G(p, a)$. As it is homogenous in p, it can be written $G(p, a) = pg(a)$ where $g(a) \equiv G(1, a)$. $G(p, a)$ must satisfy Bellman equation $EdG = rGda$ which implies, by Itô's lemma, that $g(a)$ satisfies $\delta g(a) - g_a(a) = 0$. Then $g(a) = be^{\delta a}$, where b is a constant to be determined using the value-matching and smooth-pasting conditions as

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[14] We thank an anonymous referee for pointing out these elements and their implications.

follows. At harvest age a_w and for any price p, $G(p, a_w) = V(a_w)p$, $G_p(p, a_w) = V(a_w)$, and $G_a(p, a_w) = V_a(a_w)p$, implying $g(a_w) = V(a_w)$ and $g_a(a_w) = V_a(a_w)$. Consequently,

$$\begin{aligned} G(p,a) &= be^{\delta a}p, && \text{(3a)} \\ b &= V(a_w)e^{-\delta a_w}, && \text{(3b)} \\ \frac{V_a(a_w)}{V(a_w)} &= \delta. && \text{(3c)} \end{aligned}$$

where a_w is determined implicitly by the last equation as a function of δ which is independent of price. In particular when the expected value of timber is constant ($\mu = 0$), the optimum harvest age is given by condition $\frac{V_a(a_w)}{V(a_w)} = r$, which is Wicksell's rule in the absence of uncertainty (Willassen, 1998).

To examine the effect of the existence of the second tree species, note similarly:

$$\begin{aligned} G'(p',a) &= b'e^{\delta' a}p', && \text{(4a)} \\ b' &= V'(a'_w)e^{-\delta' a'_w}, && \text{(4b)} \\ \frac{V'_a(a'_w)}{V'(a'_w)} &= \delta'. && \text{(4c)} \end{aligned}$$

In particular, when the age of the trees is zero, the stand value is either bp or $b'p'$, depending on the species. If the species is yet to be chosen, the stand value is $\max_T \left[be^{-rT}Ep_T, b'e^{-rT}Ep'_T\right]$, where T is the date at which the chosen species will be established. As only one harvest is possible in Wicksell's problem and the land has no value further to that harvest, this is also the land value:

$$F(p, p') = \max_T \left[be^{-\delta T}p, b'e^{-\delta' T}p'\right]. \tag{5}$$

The land value thus reflects the option to establish a stand of either Species P or Species P' at some optimal time. Since $F(p, p')$ is homogenous in (p, p') it can be written $F(p, p') = pf(\theta)$ where $\theta = \frac{p'}{p}$. Suppose that the land is currently bare and that it is optimal to wait rather than establish a new crop at the current time. Then there exists a time interval dt during which $F(p, p')$ satisfies Bellman Equation $EdF = rFdt$. This defines the continuation region in the plan (p, p'). Itô's lemma implies that $f(\theta)$ satisfies:

$$\frac{\overline{\sigma}^2}{2}\theta^2 f_{\theta\theta}(\theta) + \overline{\mu}\theta f_\theta(\theta) - \delta f(\theta) = 0. \tag{6}$$

with $\overline{\sigma}^2 \equiv \sigma^2 - 2\rho\sigma\sigma' + \sigma'^2$ and $\overline{\mu} \equiv \mu' - \mu = \delta - \delta'$.

This differential equation has solutions (*i.e.* the continuation region exists) because under Assumption (1) $\overline{\sigma}^2 = (\sigma - \sigma')^2 + (1-\rho)\,\sigma\sigma' > 0$ so that its characteristic equation $\frac{\overline{\sigma}^2}{2}\beta(\beta-1) + \overline{\mu}\beta - \delta = 0$ is quadratic, and its determinant $(\overline{\mu} - \frac{\overline{\sigma}^2}{2})^2 + 2\overline{\sigma}^2\delta$ is strictly positive. Thus, the characteristic equation has two distinct roots $\beta_1 > 1$ and $\beta_2 < 0$. In that case,

$$f(\theta) = b_1\theta^{\beta_1} + b_2\theta^{\beta_2}, \tag{7}$$

where b_1 and b_2 are two constants established in the proof of Proposition 2. The option value function is then $F(p, p') = pf(\theta)$ in the continuation region. However, at some levels of θ, it is optimal to plant rather than wait. Precisely:

Proposition 2 *(One rotation) When one of two tree species can be established for a single harvest only, the optimal decision is to leave the land bare unless the relative timber price is outside of an interval* $\left[\theta^*,\ \theta^{'*}\right]$. *The first time that the relative price reaches* $\theta^{'*}$ *from below (resp.* θ^* *from above) species* P' *(resp.* P*) should be established. Once species* P *(resp.* P'*) has been established the stand should be allowed to grow until it reaches its optimal harvest age defined by Formula* (3c) *(resp.* (4c)*), which is independent of the relative price. As long as the land is bare, its value is* $pf(\theta)$ *given by* (7)*; once populated with age* a *trees of species* P *(resp.* P'*), the land is worth* $be^{\delta a}p$ *(resp.* $b'e^{\delta' a}p'$*) where* b *is given by* (3b) *(resp.* b' *is given by* (4b)*).*

Proof. See the Appendix for details not in the main text. ■

The decision to choose and plant one particular species is postponed until timber prices have differentiated themselves clearly enough. This hysteresis leaves the land bare despite the fact that it would be unambiguously optimal to plant any of the two species immediately if the option to establish the other one was not available. To our knowledge, this classical result was first established by Geltner et al. (1996) in the context of a real estate land use decision. In situations where the returns expected from an asset, while high enough to warrant an immediate decision in isolation, might in the future be overtaken by the returns from an alternative asset, using such a decision rule reduces the probability of finding out *ex post* that the least profitable option was chosen, irreversibly. Once the preferred species has been established though, the asset value, hence the harvest decision, become independent of the price of the species left aside.

Clearly the hysteresis affecting species choice may occur under any parameter combination satisfying Assumption 1, whether the drifts of the price processes are identical or not, and whether or not the price of the chosen species is subject to uncertainty. In the latter case, uncertainty on the alternative asset creates an option on the opportunity cost and results in hysteresis despite the fact that the chosen asset may not be subject to uncertainty.

4. Faustmann's problem with two tree species

In order to keep the notation simple, some functions used in Section 3. will be redefined to account for multiple rotations; otherwise the notation is unchanged. Multiple harvests is the realm of Faustmann and his followers, who initially considered successive harvests of a single species whose price was both certain and constant. The problem is to find the age at which it is optimal to harvest the trees. It has been shown in the single-species literature that the optimal harvest age is independent of the price under the cost assumptions made above, whether that price is stochastic or constant. When two species are available and their prices are stochastic, we will show that the optimum age depends on the species in place and on the relative price of both species. Furthermore, we will treat separately the decision to harvest and the decision to establish the same, or the alternative, species. We start by restating some existing results with one single species. For details, see Amacher et al. (2009).

When only one species P is available, the expected value of harvesting after some time s a forest stand whose current age is a, and then replanting and harvesting the same species for an arbitrary number of rotations, is

$$G(p,a) = \max_{s} E e^{-rs}\left[p_s V(a+s) + F(p_s)\right],$$

where the land value $F(p)$ solely depends on the current price of the single species available:

$$F(p) = \max_s E e^{-rs} \left[p_s V(s) + F(p_s) \right]. \tag{8}$$

Thus, the land value coincides with the stand value at age zero as implied by the assumption that there are no planting or other investment costs; that is $F(p) = G(p, 0)$.

When the timber price is constant, the optimal harvest age a_F is a constant, known as the Faustmann rotation that is determined by Faustmann's rule (1849); it is independent of the timber price and implicitly given by:

$$\frac{V_a(a_F)}{V(a_F)} = \frac{r}{1 - e^{-ra_F}}. \tag{9}$$

The analysis has been extended to situations where the price may change over time (Brazee and Mendelsohn, 1988; Clarke and Reed, 1989; Morck et al. 1989; Thomson, 1992; Reed, 1993). When the price is governed by ($1a$) and there are no fixed costs such as planting or management costs the solution is equivalent to the deterministic solution with the timber value increasing at a constant exponential rate as established by Newman et al. (1985).[15] The following result then holds:

Lemma 1 *The land value $F(p)$ and the value of the stand at age a, $G(p, a)$, are homogenous of degree one in the timber price; the optimal harvest age is independent of that price. Precisely,*

$$G(p, a) = c e^{\delta a} p, \tag{10a}$$

$$F(p) = cp, \tag{10b}$$

$$c = \frac{V(a_f)}{e^{\delta a_f} - 1}, \tag{10c}$$

$$\frac{V_a(a_f)}{V(a_f)} = \frac{\delta}{1 - e^{-\delta a_f}}, \tag{10d}$$

where a_f is the optimal harvest age.

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[15] Saphores (2003) generalizes Faustman's formula to partial or total harvests in the case of a biomass whose stochastic growth is stock-dependent and represents the sole source of uncertainty. The optimal biomass at which harvest should occur is not a monotonic function of uncertainty, which implies that the relationship with Faustmann's rotation is ambiguous.

Proof. See the appendix ■.

Both the land value and the forest value $G(p, a)$ depend on the current price of the unique tree species linearly. The optimal single-species harvest age a_f is constant from one harvest to the next and independent of price. We call it the generalized Faustmann age in this paper. It is equal to the Faustmann (constant price) rotation if and only if the drift of the timber price process is zero; it is smaller than the Wicksellian one-rotation optimal harvest age given by $(3c)$ as $\frac{V_a(a)}{V(a)}$ is decreasing in a.

4.1 Value functions and basic concepts

We now turn to a situation not previously considered in the forestry literature and that goes beyond the cases considered in the repeated real option and replacement literatures because because it gives rise to options and value functions that ultimately depend on two, rather than one, everchanging state variables.[16] Two tree species P and P' are available. The manager has the option to harvest and to replant any combination of species, immediately after the harvest or after any delay, and to harvest again. The process goes on forever as establishing a new stand opens up the option to harvest. As in the case of a single harvest, it can be shown by adapting the proof of Proposition 1 that, to an expected value maximizer, a forest diversification strategy involving growing two species simultaneously would be dominated by one consisting in establishing one single species if at all.[17]

Thus suppose that the problem starts with trees of either species established on the forest tract. The manager may choose (i) to wait and observe the stand growing up, or (ii) to harvest the stand and replant the same species immediately, or (iii) to harvest the stand and plant the alternative species, or finally (iv) to harvest the stand and wait

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[16] The problem will be time autonomous but the value functions involved will depend on the age of the trees in place (which follows time except at discontinuities coinciding with harvests) and on the relative price of the two available species.

[17] The caveat discussed below Proposition 1 of course applies: If the growth functions are not independent of each other, mixing species may be used as a way to postpone the final species choice. If so, the options available at harvest are not actually mutually exclusive and mixing species on the land tract may be a valuable strategy. Similarly, Assumption 2 implies that there is no positive effect on growth from holding trees of various ages on a given forest tract; consequently it is optimal to harvest all trees simultanously if at all, given that the tract is initially populated with evenaged trees.

before planting one or the other species. As before we assume that there is no fixed cost associated with harvesting and planting, so that the role of the opportunity cost associated with the alternative species is highlighted.

Consider the last possibility. After harvesting, the forest manager may wait before establishing a new crop. Then the land remains bare,[18] allowing the manager to wait until timber prices evolve in such a way that it is easier to choose the right species. No gain can be achieved by using that strategy, though. Indeed suppose the manager finds it optimal to establish one species at some date t strictly posterior to the harvest. Had she planted that species immediately after harvest, she would then be better off at t because the trees would have grown already. Had she planted the other species, she could cut the trees and plant the preferred species at no cost at date t. In that case she would either be as well off, if cutting the existing trees produced no income, or she would be better off if cutting the existing trees produced some income. This proves the following proposition, which implies that only the first three possible decisions outlined in the past paragraph need be considered.

Proposition 3 *It is optimal to establish a new crop immediately after harvesting.*

This result contrasts with the one indicating that it is optimal, in the Wicksellian two-species single harvest case, to delay planting when the species are not clearly differentiated. Indeed, the decision to choose a species that turns out to be undesirable *ex post* does not have any opportunity cost in case of multiple rotations when management or replacement costs are inexistent: trees that do not turn out to remain desirable after planting may be cut and replaced with the species that turns out to be desirable, a possibility that does not arise in the single rotation case where the initial choice is final.

As mentioned already, this result crucially depends on management or replacement costs being zero. Were they strictly positive, the cost of choosing the wrong species would not be zero as these costs would have to be reincurred in order to change species.

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[18] Allowing for costs of keeping the land bare, such as weeding or protection against erosion, would only reinforce the result.

However this would not be necessarily enough to justify leaving the land bare after harvesting. An alternative possibility not present in the one-shot case would be to postpone harvest. As long as it remains less costly to hold standing trees than to cut and sell the harvest, it will be preferable to postpone the harvest rather than leave the land bare. Nonetheless, if the two species prices remain insufficiently differentiated for too long, the growth of standing trees will slow down to a trickle that will no longer justify the opportunity cost of postponing the harvest. Trees will be cut and the land will remain bare for a while as in the single harvest case.

To recap, other things equal, comparing a once and for all choice between two alternatives species with indefinitely repeated choices between these alternatives reveals that inaction, taking the form of leaving the land bare, is less probable in the later case and is never optimal if the exercise of the option has no cost other than an opportunity cost of not choosing the alternative species. A related result has been found for indefinitely repeated investment options by Malchow-Møller and Thorsen (2005) who showed that the value of waiting arising from an option to invest in a piece of equipment is lower when the option is repeated than when it is a single option. These authors treated the aquisition of the new equipment as costly in contrast with the zero replacement cost assumption made here. However their model of equipment replacement did not provide for equipment depreciation: Unlike trees whose growth slows down with age, machines in Malchow-Møller and Thorsen do not loose productivity with age. As a result the sole reason to remove them from the plant is to replace them with new ones. The shop will never be empty despite the replacement cost.

Given that the land is populated with one uniform-age species, consider now alternatives $(i) - (iii)$ which require choosing the harvest age of the species in place, and the replacement species. Let $F(p, p')$ denote the value of the bare land, which is the value of the options to indefinitely exploit the forest tract, choosing the appropriate replacement species after each harvest. Let $G(p, p', a)$ (resp. $G'(p', p, a)$) denote the value of the forest (land and trees) when the forest tract is populated with trees of species P (resp. P') of age a and price p (resp. p') while the price for the other species is p' (resp. p).

Precisely,

$$\begin{aligned} G(p,p',a) &= \max_s E e^{-rs}\left[p_s V(s+a) + F(p_s,p'_s)\right], &\text{(11a)}\\ G'(p',p,a) &= \max_s E e^{-rs}\left[p'_s V'(s+a) + F(p_s,p'_s)\right]. &\text{(11b)} \end{aligned}$$

As the timber prices p and p' follow GBM's, the land value function $F(p,p')$ and the forest value functions $G(p,p',a)$ and $G'(p',p,a)$ are homogenous of degree one in (p,p') so that one can define the following functions, which will be called reduced functions in the sequel:

$$\begin{aligned} g(\theta,a) &= \frac{1}{p}G(p,p',a), &\text{(12a)}\\ g'(\theta,a) &= \frac{1}{p}G'(p',p,a), &\text{(12b)}\\ \text{and } f(\theta) &= \frac{1}{p}F(p,p'). &\text{(12c)} \end{aligned}$$

The reduced functions respectively give the land value and the forest value in terms of the numeraire p. Given $(11a)$ and $(11b)$, they satisfy:

$$\begin{aligned} g(\theta,a) &= \max_s\left\{e^{-\delta s}V(a+s) + e^{-rs}E\left[\frac{p_s}{p}f(\theta_s)\right]\right\}, &\text{(13a)}\\ g'(\theta,a) &= \max_s\left\{e^{-\delta' s}\theta V'(a+s) + e^{-rs}E\left[\frac{p_s}{p}f(\theta_s)\right]\right\}, &\text{(13b)}\\ f(\theta) &= \begin{cases} \max_s\left\{e^{-\delta s}V(s) + e^{-rs}E\left[\frac{p_s}{p}f(\theta_s)\right]\right\}, & \theta\le\theta^*,\\ \max_s\left\{e^{-\delta' s}\theta V'(s) + e^{-rs}E\left[\frac{p_s}{p}f(\theta_s)\right]\right\}, & \theta\ge\theta^*, \end{cases} &\text{(13c)} \end{aligned}$$

where θ^* is the relative price below (resp. above) which it is optimal to plant Species P (resp. Species P') on a bare land. The existence of θ^* is proven in Lemma 2. Note that $\frac{p_s}{p}$ does not depend on the current price p as it is a GBM whose current value equals one. Therefore, the optimal harvest age depends only on the current relative timber price $\theta = \frac{p'}{p}$.

As a consequence of Proposition 3, it is certain that the trees being harvested are replaced immediately. At relative prices below the switching value θ^*, species P is planted if the land is bare; above θ^*, it is optimal to plant species P' on a bare land. It is certain that θ^* exists as species P must be chosen when θ tends to zero and species P' must be chosen when θ tends to infinity. This is formally proven in the next lemma.

Lemma 2 *There exists a unique, strictly positive, value of θ, θ^* such that, if $\theta < \theta^*$ (resp. $\theta > \theta^*$) and the land is bare, it is optimal to plant species P (resp. species P') immediately, while indifferently planting P or P' on bare land is optimal if $\theta = \theta^*$.*

Proof. The set of $\theta > 0$ for which it is optimal to plant P is not empty as for θ positive and sufficiently small it is optimal to plant P. This set is bounded as for θ sufficiently high it is optimal to plant P'. Being not empty and bounded, the set of $\theta > 0$ for which it is optimal to plant P has a finite maximum θ^*. The uniqueness of θ^* results from continuity. ■

Both the forest value and the land value functions increase when the price of either species increase. As a matter of fact, if the land is populated with one species and the price of the other species increases while the price of the species in place does not change, the forest value increases, because the exercise of the option to plant the other species becomes more likely. Furthermore, when relative prices are such that a switch to the alternative species might be a profitable decision, one expects that the value functions should be higher than if the possibility to switch was not available. On the contrary, when relative prices are extreme, the land value should reflect the expected present value of the revenues associated with establishing and harvesting the same species forever, pc and $p'c'$ for species species P and P' respectively. Formally,

Lemma 3 *The reduced land value function is such that*

$$f(\theta) = g(\theta, 0) = g'(\theta, 0). \tag{14}$$

Furthermore, $f(\theta)$ is increasing in θ with $\lim_{\theta \to 0} f(\theta) = c$, $\lim_{\theta \to +\infty} f(\theta) = c'\theta$, where $c = \frac{V(a_f)}{e^{\delta a_f} - 1}$ and $c' = \frac{V'(a'_f)}{e^{\delta' a'_f} - 1}$.

Proof. The land value function $F(p, p')$ must be strictly increasing in p; then $F_p(p, p') > 0$. As $F_p(p, p') = f_\theta(\theta)$ then $f_\theta(\theta) > 0$ and therefore $f(\theta)$ is strictly increasing in θ. When θ tends to zero, that is when p' tends to zero while p is strictly positive with $p' \ll p$, P' is to remain the preferred species and the problem collapses to the single species case. Therefore, using (10b), $\lim_{\theta\to 0} pf(\theta) = cp$ or $\lim_{\theta\to 0} f(\theta) = c$. A similar proof shows that $\lim_{\theta\to+\infty} f(\theta) = c'\theta$. (14) is true because the definitions of g and f (respectively g' and f) coincide when $a = 0$. ■

4.2 Choosing between alternative replacements repeatedly: the solution

Equations (12a) and (12b) formally define the forest value in terms of the numeraire p (respectively when Species P or Species P' are in place) as real options. That is to say, their value results from the possibility, but not the obligation, to exercise the option to harvest after an optimally chosen laps of time s. Leaving aside well known issues concerning the market environment in which the problem is framed,[19] this harvest (or retirement) option is not a standard American option. As with an American option, the boundary is free and the expiry period infinite. However, with an American option the value of the underlying asset is observable; here the value of the underlying asset consists of two elements. The first one is the value of the timber being harvested; this is observable. The second one is the value of the land $pf(\theta)$; this is endogenous and can be determined only once the optimal harvest rule is solved for.

The solution of the problem is a harvest rule combined with a species choice rule indicating the action required according to the age of the trees in place and to the relative price. Thus we need to characterize the continuation region when species P is in place, and the continuation region when species P' is in place. In either case the continuation region is the set of points (θ, a) in $\mathbb{R}^+ \times \mathbb{R}^+$ where it is optimum to wait rather than exercise the option of harvesting and choosing a replacement species.[20] Suppose that species P is in place, a situation which is desirable at relatively low values of θ. Clearly there is an age above which it is desirable to cut the trees so that the continuation region, if it exists, is bounded above (along the a dimension). Let $a^+(\theta)$ denote the upper boundary of the continuation region, the age above which it is optimum to harvest and below which (at least for a small age variation) it is desirable to allow

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[19] For conditions under which the dynamic programming formulation used here gives the same valuation as the contingent claims analysis pioneered by Black and Sholes (1973), see Chapter 4 of Dixit and Pindyck (1994).

[20] While θ is the same variable in both continuation regions, a is specific to the species in place.

the trees to grow further, given θ. If it exists, it is the smallest value of a such that immediate harvesting is optimum, that is:

$$\arg\max_{s}\left\{e^{-\delta s}V\left(a+s\right)+e^{-rs}E\left[\frac{p_s}{p}f\left(\theta_s\right)\right]\right\}=0. \tag{15}$$

At high enough values of θ, the alternative species P' becomes so attractive that there is no age at which it would be desirable to allow trees of species P to grow any further. At such high values of θ, the continuation region of species P does not exist. Let $\overline{\theta}$ denote the value of θ above which the continuation region does not exist for any harvest age and below which the continuation region exists for some age. Thus $\overline{\theta}$ signals the right-hand end of the continuation region. It can be shown by contradiction that this region exists for any $\theta<\overline{\theta}$. $\overline{\theta}$ is strictly higher than θ^*. Indeed, at $\theta=\theta^*$ the decision maker is indifferent between stands of either species P or P' populated with trees of age zero, which take periods of $\underline{a}$ or $\underline{a}'$ before producing any valuable timber. Then if the stand is populated with trees of species P whose age is strictly between 0 and $\underline{a}$ while $\theta=\theta^*$, it is preferable to let them reach some age above $\underline{a}$, possibly allowing θ to become greater than θ^* before cutting them. This proves that $\theta^*\leq\overline{\theta}$. This simple argument also implies that, for values of the relative price in $\left[\theta^*,\overline{\theta}\right]$, it is optimal to allow the trees to continue growing if they are older than some minimum age. Consequently, the continuation region has a lower boundary $a^-\left(\theta\right)\geq 0$ for $\theta\in\left[\theta^*,\overline{\theta}\right]$.

When Species P is in place, the lower boundary $a^-(\theta)$ and the higher boundary $a^+(\theta)$ satisfy the conditions:[21]

$$\text{For }\theta\in\left[\theta^*,\overline{\theta}\right],\ \begin{array}{l} g\left(\theta,a\right)>V\left(a\right)+f(\theta),\ \text{if } a^-(\theta)<a<a^+(\theta),\\ g\left(\theta,a\right)<V\left(a\right)+f(\theta),\ \text{if } a<a^-(\theta)\text{ or } a>a^+(\theta),\\ g\left(\theta,a\right)=V\left(a\right)+f(\theta),\ \text{if } a=a^-(\theta)\text{ or } a=a^+(\theta).\end{array} \tag{16}$$

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[21] A similar analysis applies to the continuation region when species P' is in place. Remembering that $\theta'^*\equiv\theta^*$, it follows that $\overline{\theta}'\leq\theta^*$ and that the continuation region for P' has upper and lower boundaries $a'^+(\theta)$ and $a'^-(\theta)$ on $\left[\overline{\theta}',\theta^*\right]$, and has only an upper boundary $a'^+(\theta)$ on $[\overline{\theta}',+\infty[$. While the relative price θ is the same variable in both cases, only considered at different values, one should note that the expressions corresponding to each region are not the mirror image of each other, although the analysis obeys a symmetric logic. For example, obviously, $\overline{\theta}'\leq\theta^*\leq\overline{\theta}$ as species P' is desirable at high relative prices while the continuation region for P' does not exist below $\overline{\theta}'$ by definition of $\overline{\theta}'$. Also, the relevant value functions $(13a)$, $(13b)$ and $(13c)$ are not identical at values of θ on either side of θ^*, implying differences in the continuation regions.

For $\theta \in \left[\theta^*, \overline{\theta}\right]$ and $a \in]a^-(\theta), a^+(\theta)[$, the condition $g\left(\theta, a\right) > V\left(a\right) + f(\theta)$ indicates that, if the land is populated with trees of species P and age a, it is preferable to allow them to reach maturity and harvest them when they reach age $a^+(\theta)$, rather than harvesting them immediately and planting species P' to obtain $pV\left(a\right)$ plus land value $pf(\theta)$. For $\theta \in \left[\theta^*, \overline{\theta}\right]$ and $a < a^-(\theta)$, the trees are too young to be allowed to reach maturity: harvesting immediately and planting the alternative species yields a higher expected value than pursuing the objective in (15); if $a > a^+(\theta)$, a has been allowed to exceed the age, at which (15) is satisfied; such situation should not occur if the harvest age is chosen optimally but may characterize the initial state of the problem. Condition $g\left(\theta, a\right) < V\left(a\right) + f(\theta)$ then indicates that it is optimal to harvest immediately and switch to Species P'.

In the continuation region, *i.e.* for (a, θ) such that $0 \leq a \leq a^+(\theta)$ and $0 \leq \theta \leq \theta^*$ or $a^-(\theta) \leq a \leq a^+(\theta)$ and $\theta^* \leq \theta \leq \overline{\theta}$, the forest value function $G(p, p', a)$ should satisfy Bellman equation $E\left(dG(p, p', a)\right) = rG(p, p', a)da$ which can be written as the following partial differential equation governing the reduced forest value function $g(\theta, a)$ in the continuation region (proof in the appendix):

$$\frac{\overline{\sigma}^2}{2}\theta^2 g_{\theta\theta} + \overline{\mu}\theta g_\theta - \delta g + g_a = 0. \tag{17}$$

On the upper boundary $a^+(\theta)$ of the continuation region, the following value matching and smooth pasting conditions apply[22]

$$g\left(\theta, a^+\left(\theta\right)\right) = V\left(a^+\left(\theta\right)\right) + f\left(\theta\right), \tag{18a}$$

$$g_\theta\left(\theta, a^+\left(\theta\right)\right) = f_\theta\left(\theta\right), \tag{18b}$$

$$g_a\left(\theta, a^+\left(\theta\right)\right) = V_a\left(a^+\left(\theta\right)\right). \tag{18c}$$

The smooth pasting condition can be written in either of the two forms (18*b*) or (18*c*); given the value-matching condition these two forms are equivalent.

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[22] The lower boundary $a^-(\theta)$ is not characterized by any analog of the smooth pasting condition because, unlike the upper boundary, it does not satisfy any maximizing condition similar to (15); however the third equation in (16) may be interpreted as a value matching condition when $a = a^-(\theta)$.

Similarly, one can show that the reduced forest value function $g'(\theta, a)$ should satisfy the same partial differential equation as $g(\theta, a)$, that is

$$\frac{\overline{\sigma}^2}{2}\theta^2 g'_{\theta\theta} + \overline{\mu}\theta g'_\theta - \delta g' + g_a = 0. \tag{19}$$

The boundary conditions are slightly different despite the symmetry. On the upper boundary $a^{+\prime}(\theta)$ of the corresponding continuation region, the following value matching and smooth pasting conditions apply:

$$\begin{aligned} g'\left(\theta, a^{+\prime}(\theta)\right) &= \theta V'\left(a^{+\prime}(\theta)\right) + f(\theta), && \text{(20a)} \\ g'_\theta\left(\theta, a^{+\prime}(\theta)\right) &= V'\left(a^{+\prime}(\theta)\right) + f_\theta(\theta), && \text{(20b)} \\ g'_a\left(\theta, a^{+}(\theta)\right) &= \theta V'_a\left(a^{+\prime}(\theta)\right). && \text{(20c)} \end{aligned}$$

We are now a short step from a description of the main analytical properties of the solution, i.e. the harvest decision rule as function of tree age and relative prices defined by the upper and lower boundaries. In fact this description is contained in expressions (17) - (18*c*) and their counterparts (19) - (20*c*) applying when Species P' is in place. These expressions involve the unknown functions g, g', and f. Eliminating g one has Equation (21) below, which links the function $f(\theta)$ to the harvest age on the upper boundary $a^+(\theta)$. A similar equation, (22) below, applies when species P' is in place. Finally eliminating $\frac{\overline{\sigma}^2}{2}\theta^2 f_{\theta\theta}(\theta) + \overline{\mu}\theta f_\theta(\theta)$ from (21) and (22)[23] yields Equation (23) which

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[23] One can show that the reduced forest value function $g'(\theta, a)$ should satisfy the same partial differential equation as $g(\theta, a)$, that is

$$\frac{\overline{\sigma}^2}{2}\theta^2 g'_{\theta\theta} + \overline{\mu}\theta g'_\theta - \delta g' + g_a = 0$$

The boundary conditions are slightly different despite the symmetry. On the upper boundary $a^{+\prime}(\theta)$ of the corresponding continuation region, the following value matching and smooth pasting conditions apply:

$$\begin{aligned} g'\left(\theta, a^{+\prime}(\theta)\right) &= \theta V'\left(a^{+\prime}(\theta)\right) + f(\theta) \\ g'_\theta\left(\theta, a^{+\prime}(\theta)\right) &= V'\left(a^{+\prime}(\theta)\right) + f_\theta(\theta) \\ g'_a\left(\theta, a^{+}(\theta)\right) &= \theta V'_a\left(a^{+\prime}(\theta)\right) \end{aligned}$$

applies on the interval $\left[\overline{\theta}', \overline{\theta}\right]$ where both continuation regions exist and is written in terms of the growth functions V and V' only:

$$\frac{\overline{\sigma}^2}{2}\theta^2 f_{\theta\theta}(\theta) + \overline{\mu}\theta f_\theta(\theta) + V_a\left(a^+(\theta)\right) = \delta V\left(a^+(\theta)\right) + \delta f(\theta), \ \theta \in \left[0, \overline{\theta}\right] \tag{21}$$

$$\frac{\overline{\sigma}^2}{2}\theta^2 f_{\theta\theta}(\theta) + \overline{\mu}\theta f_\theta(\theta) + \theta V'_a\left(a'^+(\theta)\right) = \theta\delta' V'\left(a'^+(\theta)\right) + \delta f(\theta), \ \theta \in [\overline{\theta}', +\infty[\tag{22}$$

$$\delta V\left(a^+(\theta)\right) - V_a\left(a^+(\theta)\right) = \theta\left[\delta' V'\left(a'^+(\theta)\right) - V'_a\left(a'^+(\theta)\right)\right], \ \theta \in \left[\overline{\theta}', \overline{\theta}\right] \tag{23}$$

Equations (21) and (22) are non arbitrage conditions; using p as numeraire, the right-hand side is the opportunity cost of waiting to harvest, per time unit; the first two terms on the left-hand side give the expected change in land value due to the variation of θ occurring during the same time unit, where the presence of a second derivative arises from the stochasticity of θ; and the third term on the left-hand side is the increase in the crop value due to timber growth per time unit. Clearly the land value is the same whether Species P or Species P' is in place; consequently (21) and (22) differ only in what concerns the species in place: terms involving the growth functions V and V', but also the relative value θ of the wood being produced. Equation (21) ensures the consistency of the harvest rules when either Species P or Species P' may be in place, that is to say in the interval $\left[\overline{\theta}', \overline{\theta}\right]$. Since the bare land is the same in both cases, the consistency condition only affects the trees: it says that the financial opportunity cost of holding standing timber, net of the value of physical growth, should be the same for both species.

The properties of the upper and lower boundaries that can be derived from the above analysis are spelled out in the following proposition for Species P and illustrated in Figure 1 for both Species P and Species P'.

Proposition 4 *Suppose that species P is currently in place. The upper boundary $a^+(\theta)$ is such that $a^+(0) = a_f$; it is continuous on $\left[0, \overline{\theta}\right]$, non increasing at low values of θ, strictly increasing on $\left[\overline{\theta}', \theta^*\right]$, and strictly decreasing on $\left[\theta^*, \overline{\theta}\right]$. The lower boundary $a^-(\theta)$ is continuous and strictly increasing on $\left[\theta^*, \overline{\theta}\right]$, with $a^-(\theta^*) = 0$. The upper and lower boundaries meet at Point $(\bar{\theta}, \underline{a})$. The properties of the boundaries $a^{+'}(\theta)$ and*

$a^{-'}(\theta)$ *that apply when Species* P' *is in place follow by symmetry.*

Proof. See the appendix. ■

Figure 1 describes the harvest decision and species replacement rule according to the species in place, the age of the stand, and the current relative price of the two species available to replace the trees of the current harvest. The vertical axis gives the age a of the current stand which consists of trees of either Species P or Species P'. Any state of the problem is represented by a pair (θ, a) and the information about which species is in place. Over time, such pair may move to the right or to the left as θ follows a GBM of drift and volatility $\bar{\mu}$ and $\bar{\sigma}$ (defined below Equation (6)); it may move only up as the age of trees can only increase. When trees are cut, say at a relative price of θ, the state of the problem jumps down to $(\theta, 0)$ as a new stand of trees aged 0 is immediately established; if $\theta < \theta^*$ ($\theta \geq \theta^*$), the new stand is of Species P (Species P'). For any pair (θ, a), there is an optimum decision (harvest or wait) according to the position of the pair relative to the appropriate boundary; the relevant boundary is the continuous curve if Species P is in place; the dashed curve applies if the stand consists of trees of Species P'.

When the current stand consists of trees of Species P, the optimal rule is: if $\theta < \theta^*$, harvest if the age of the trees is or exceeds $a^+(\theta)$ (i.e. if the pair (θ, a) is above the continuous curve); otherwise wait and allow the trees to grow further; if $\theta \geq \theta^*$, harvest if the (θ, a) pair is to the right of either $a^+(\theta)$ or $a^-(\theta)$; otherwise wait and allow the trees to grow further. When the current stand consists of trees of Species P', the curves that define the optimal rule are $a'^+(\theta)$ and $a'^-(\theta)$; harvesting is optimal above $a'^+(\theta)$ and to the left of $a'^-(\theta)$.

The dynamics of the solution deserve further comments. Suppose that Species P is in place. Since age can only increase, a pair (θ, a) such that $\theta < \bar{\theta}$ and $a < a^+$ is likely to cross the $a^+(\theta)$ boundary from below after some time, signalling financial maturity and the need to harvest. However, if a is initially low enough, the pair may never reach the

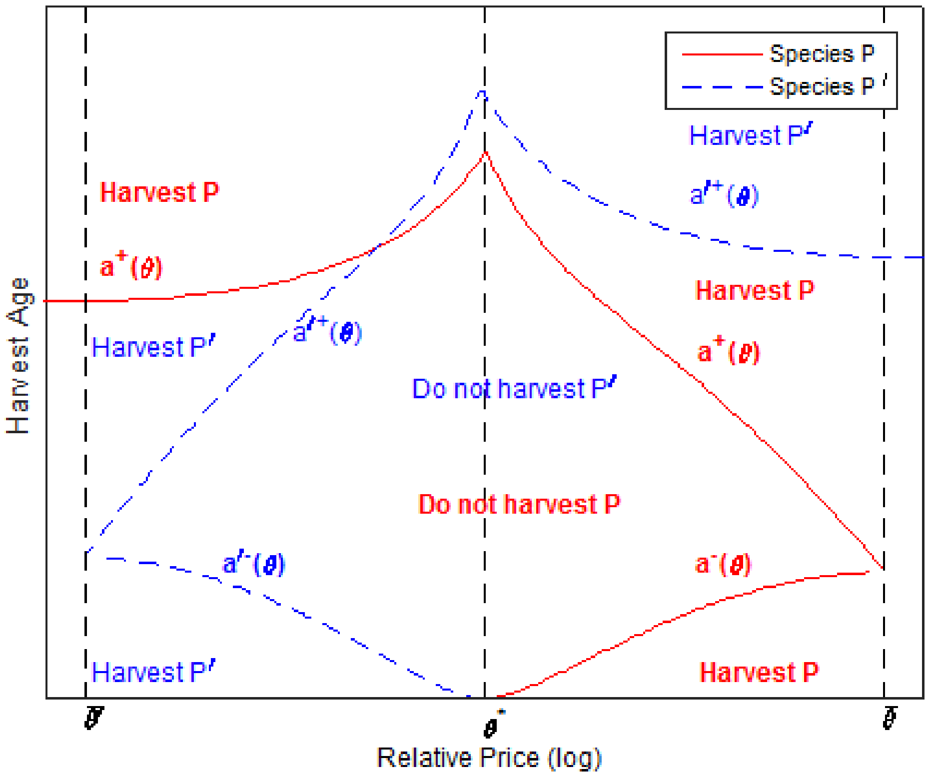

Figure 1: Boundaries of the Continuation Regions for Species P and P' (On a bare land, P (resp. P') is planted on the LHS (resp. RHS) of θ^*).

$a^+(\theta)$ boundary and instead cut the $a^-(\theta)$ boundary from the left. This would happen if the relative price increased very fast, making Species P' so desirable that allowing current trees of Species P to reach maturity did not match the expected benefit from replacing them immediately with trees of Species P' and age zero. When the absorbing barrier $a^-(\theta)$ is reached, the system is reset to state $(\theta, 0)$ with the introduction of P' trees. This implies that states such that $a < a^-(\theta)$ with $\theta^* < \theta < \bar{\theta}$ cannot be reached by any optimal trajectory while Species P in place. Such states can only be considered as initial values, not as inherited from any previous series of rational decisions.

At high values of θ, Species P' is relatively more attractive than Species P. In fact, if $\theta \geq \bar{\theta}$ a P stand should be harvested immediately and replaced with P' trees whatever the age of the current stand. Similarly, if $\theta \leq \bar{\theta}'$, a P' stand should be immediately replaced with P trees. Again, such states can only be considered as initial values, not as inherited from any previous series of rational decisions. Once the appropriate Species is in place, it should be managed optimally as just described. The rule is simple but differs notably from the single species rule. Not only is it possible that trees need to be harvested before reaching maturity (if boundaries a^- or a'^- are reached) but the optimum harvest age at maturity is not constant as in the case of a single species; it depends on θ and only tends toward the single species generalized Faustmann age a_f when $\theta \rightarrow 0$ (Species P) or a'_f when $\theta \rightarrow \infty$ (Species P').

Furthermore, while Proposition 4 indicates that the optimum harvest age does not exceed a_f at low θ values and later increases to reach a peak at θ^*, it is not clear whether or not it exceeds a_f at intermediate values $\theta < \theta^*$. As a matter of fact the $a^+(\theta)$ curve under certainty repeatedly cuts the horizontal line $a = a_f$ on $[0, \theta_0]$ as depicted in Figure 2. By continuity, it is then certain that the $a^+(\theta)$ curve also cuts the $a = a_f$ line more than once at low levels of volatility.

The monotonicity properties of $a^+(\theta)$ and $a^-(\theta)$ stated in Proposition 4 are independent of the parameters of the price processes or the species growth functions. Therefore, by symmetry, if species P' is in place then $a'^+(\theta)$ is strictly increasing while the lower boundary $a'^{-}(\theta)$ is strictly decreasing on $\left[\overline{\theta}', \theta^*\right]$ as depicted. Both curves $a^{+}(\theta)$ and $a'^{+}(\theta)$ reach a strict maximum at θ^*. This result has a clear economic interpretation. When the relative price is close to θ^* the risk of choosing the 'wrong' species if harvest takes place is high because chances are high that the relative price will cross θ^*. Such a reversal would make it desirable to choose the alternative species if the land was bare, which can be done at no direct cost. However harvesting entails the opportunity cost of foregoing the value of further growth by the mature trees. This opportunity cost of harvesting dominates the benefits from early maturity of the new crop if the latter is not likely enough to reach maturity. This explains why the optimum harvest age has a peak at θ^*. It is also clear why $a^{+}(\theta)$ should be decreasing beyond θ^* ($a'^{+}(\theta)$ should be increasing when θ approaches θ^* from below): suppose that the species in place is P (so that the relevant boundary is $a^{+}(\theta)$) and consider values of θ above θ^*; then P is less desirable the further θ is from θ^*; hence the trees should be cut younger, the further away θ is from θ^*, not only because the possibility of making a mistake becomes more remote, but also because the opportunity cost of not establishing the desirable species becomes higher.

4.3 Volatility

In order to describe the solution beyond its qualitative characteristics, it is necessary to use numerical methods[24]. The reduced forest value function $g(\theta, a)$ must satisfy the partial differential equation (17) and the value-matching and smooth pasting conditions (18a), (18b), and (18c). The optimal stopping problem of valuing the forest when species P is in place is similar to the problem of valuing an American option with free boundary. However, because the free boundary is not known in advance, the value-matching and smooth pasting conditions cannot be of immediate help. These conditions can be used to localize the free harvest boundary once the land value is determined. We do so by

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[24] To isolate the role of the stochastic price processes defined by (1a) and (1b), we use the same volume growth function for both species to generate numerical illustrations in this section: $V(a) = V'(a) = V_\infty \left(1 - e^{-\alpha(a-\underline{a})}\right)$ where $V_\infty = 100$ is the timber volume when the age tends to infinity, $\alpha = 0.01$, and $\underline{a} = 10$ is the minimum age for positive growth. The discount rate r is 5 %. For the stochastic processes (1a) and (1b), $\mu = 0.01$, $\mu' = 0.04$, and $\sigma = \sigma' = 0.1$.

successive iterations as explained in Appendix B.

Figures 2 and 3 respectively compare the management rule (harvest age and species choice) and the land value functions when at least one timber price is stochastic and when both are deterministic.[25] The price drifts are identical in both cases, but the volatility of θ is zero under certainty, instead of $\overline{\sigma}^2 > 0$ under uncertainty. For clarity, Figure 2 only exhibits the boundaries applying when Species P is in place; the uncertainty case reproduces the boundary shown in Figure 1 for P.

The deterministic case is remarkably different from the stochastic case. We only sketch its main features here to highlight the effect of price volatility. A detailed analytical treatment is given in Ben Abdallah and Lasserre (2015). Indeed suppose that $\mu < \mu'$ and $\sigma = \sigma' = 0$; p' rises faster, or diminishes more slowly, than p. Then it is certain that Species P' will eventually become more attractive than P if this is not already the case. Hence if Species P is in place, it should be replaced by Species P' either immediately or after a certain laps of time. In fact, when the land is bare, Species P' should be established if $\theta \geq \theta_0$ under certainty, instead of $\theta \geq \theta^*$ under uncertainty, where $\theta_0 < \theta^*$ as shown in Figure 2.[26]

Thus uncertainty increases the switching threshold as in traditional real options problems: in a typical real options asset-adoption problem, the value of the project to be undertaken (adopt the new species) must exceed the level required under certainty by an amount corresponding to the value of the flexibility being given up by switching.[27] This logic explains why the relative price at which harvest is optimal on the right of θ^*, i.e. when harvesting implies switching to the alternative species, is higher than the price required at the same age under certainty.

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[25] As shown earlier, the stochasticity of one price confers a value to the option to switch even if the price of the species to be adopted is certain.

[26] Furthermore, under certainty, it is certain that the relative advantage of P' over P keeps increasing so that the switch will not be reversed. Under uncertainty there may be an indeterminate number of switches between species despite the fact that the drift of p' is higher than the drift of p. Nonetheless, choosing P' over P upon harvest implies the loss of the flexibility to choose the species of the next stand.

[27] Under uncertainty, the switch is not necessarily final. However, even if it is not irreversible, planting the wrong species involves a revenue loss.

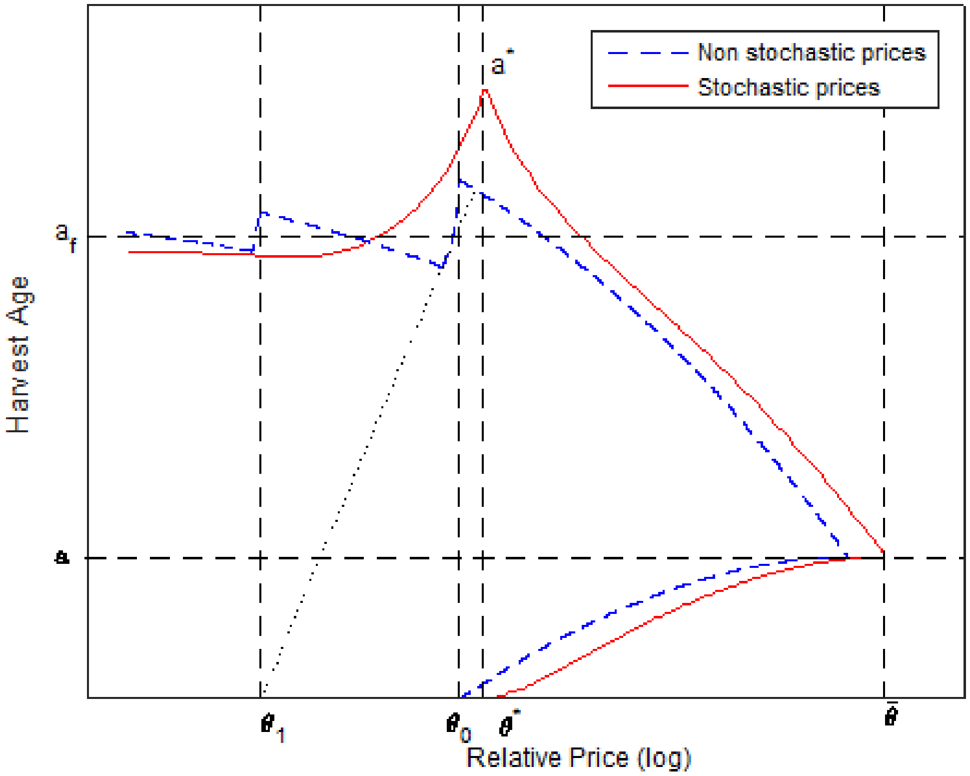

Figure 2: Harvest Age and Switching Price under Certainty and Uncertainty ($\mu < \mu'$)

For θ below the switching values θ_0 or θ^*, the situation is not as clear-cut because cutting earlier may increase flexibility rather than reduce it. Consider a harvest occurring at a relative price $\theta_1+\varepsilon$ slightly above θ_1; under certainty, as shown by the corresponding trajectory of (θ, a) given by the dotted line rising from $(\theta_1, 0)$, this implies that the next harvest will bring a switch to P' since it will occur at a price exceeding θ_0. Under uncertainty, the switch at the next harvest is quite uncertain: if the realized trajectory of (θ, a) is to the left of θ^* when it hits the harvest boundary that applies when prices are stochastic (the continuous curve), there will be no switch; if it is to the right of θ^* there will be a switch. The position of the stochastic harvest boundary below the non stochastic boundary at θ_1 means that harvesting brings a gain in flexibility (increases the value of the option to choose between P and P' at the next harvest) and thus does not require as high a price than in case of certainty. This gain in flexibility dominates other considerations at some values of θ below θ^* in the example represented by the figure, explaining why the harvest boundaries under certainty and under uncertainty may cross each other at some relative prices below the switching threshold.

When the relative price tends to zero or to infinity, that is when the land value results from an infinite succession of harvests of Species P or Species P' respectively, the optimal harvest age tends toward the single species values of α_f or a'_f respectively. Accordingly, in Figure 3, the reduced land value function tends toward its single-species levels of c when Species P is preferable (low θ) or $c'\theta$ when Species P' is to be chosen (high θ). Except at these asymptotic values of θ, the land value is strictly higher than if either species was exploited exclusively forever, as shown in the figure. If there is no uncertainty, however, the reduced land value function becomes equal to $c'\theta$ as soon as θ reaches θ_0 rather than when θ tends toward infinity. This is because, under certainty, θ can only increase, so that the switch from Species P to Species P' occurs only once. Consequently the prospect to switch from P' to P is worthless beyond θ_0, implying that the forest value then results from harvesting P' repeatedly forever.

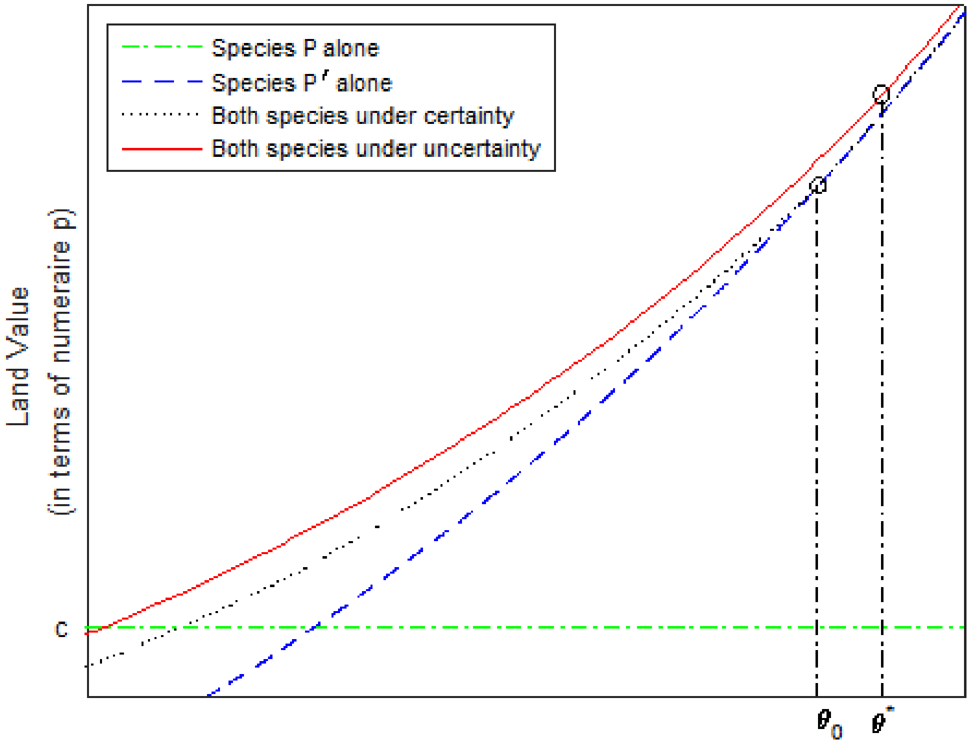

Figure 3: Land Value as Function of Relative Price ($\mu < \mu'$)

5. Conclusion

As a way to introduce the repeated alternative options problem at the core of this paper, we began with a one-rotation version of the forest management problem with two species. Starting with a bare plot of land the problem consists first in choosing the appropriate time to establish trees and then in choosing the optimal harvest age. We found that diversification (mixing the two species) is not optimal, a result crucially dependent on the assumption that the growth of one species is independent of the presence of trees of the other species. Furthermore, immediate planting is optimal only if the price of one species is sufficiently high compared to the price of the other species. Otherwise, if current prices are such that the expected value of harvesting one species is relatively close to the expected value of the alternative, it is preferable to leave the land vacant until prices evolve to make one species clearly preferable. This type of hysteresis, first identified by Geltner et al. (1996) in a urban economics context, occurs despite the fact that immediate planting would be optimal if only one of the two species were available. Indeed, to the extent that future prices are stochastic the choice involves the potential opportunity cost of establishing the wrong species. Such mistake is irreversible as the project involves only one harvest. This is why it is preferable to wait and plant only once the expected values of the alternative species at harvest are sufficiently far away from each other to make the probability of a reversal acceptably low. However, once the preferred species has been established, the asset value and the remaining harvesting decision, become independent of the price of the species left aside. Bygones are bygones.

The main contribution of the paper is when the forest land is to be exploited for an infinite number of rotations. Here one must decide whether current trees have reached harvest age or not; if so, whether they should be replaced immediately or after any optimal delay; and which species or combination of species should be chosen. As in the single harvest case, and still subject to the assumption that a species' growth is independent of the presence of trees of the other species species, diversification is not desirable. However the land should never be left vacant. This rule differs from the single harvest case because introducing many rotations, first dilutes the irreversibility effect, and second introduces the growth of existing trees among the benefits of postponing replacement. Nonetheless the probability of later regretting a choice of replacement is highest when the relative price equals the threshold that determines which species is to be preferred. Allowing the current stand to grow further is a way to delay that choice while it is too risky, which explains that the optimal harvest age is higher, the closer the relative species price to the switching threshold.

These results were obtained analytically, under the assumption that replanting or replacement costs are absent. This assumption has allowed us to focus on the key novelty explored in this paper, which is the role of the opportunity costs repeatedly incurred each time that an alternative is chosen over another one. Clearly these opportunity costs also exist when replanting or replacement costs are present, but their effect are likely to be diluted and to require numerical methods for their appraisal.

The option to postpone replacement by increasing the harvest age is exploited for all future replacements, implying that the land value is an American option with free boundary, infinite expiry period, and endogenous payoff. This is also true when the value of a firm arises from repeated retirements of depreciating assets and their replacement by alternatives whose capacity to generate value is uncertain.

6. Acknowledgements

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APPENDIX A: PROOFS

Proof of Proposition 1

Consider a risk neutral decision maker who chooses the proportion λ of a bare piece of land to be devoted to either one of the species while simultaneously choosing the planting date. The rest of the land $1-\lambda$ is devoted to the other species and planted simultaneously or later. Harvests take place at dates to be chosen optimally for each species. Assuming without loss of generality that species P is planted at T and species P' is established at T', the value of the asset is then

$$F(p,p') = E \max_{T,T',\lambda\in[0,1]} \left\{ \lambda e^{-rT} \max_{s} E_{T} \left(e^{-rs} V(s) p_{T+s} \right) + (1-\lambda)\, e^{-rT'} \max_{s'} E_{T'} \left(e^{-rs'} V'(s') p'_{T'+s'} \right) \right\} \tag{24}$$

where E is the expected value operator conditional on current information, and E_T (respectively $E_{T'}$) is the expected value operator conditional on information at T (respectively T'). Since λ and $1-\lambda$ do not affect the choices of s or s' in problem (24), and since they enter the objective function multiplicatively, they do not affect the choices of T or T'. Consequently λ can be determined given the optimal rules for choosing T and T':

$$\begin{aligned} F(p,p') \;=\; & E \max_{\lambda\in[0,1]} \left\{ \lambda \max_{T} E \left(e^{-rT} \max_{s} E_T e^{-rs} V(s) p_{T+s} \right) \right. \\ & \left. + (1-\lambda) \max_{T'} E \left(e^{-rT'} \max_{s'} E_{T'} e^{-rs'} V'(s') p'_{T'+s'} \right) \right\} \end{aligned}$$

The above problem is linear in λ. Consequently the solution is either $\lambda = 1$ or $\lambda = 0$. This establishes that diversification is not optimum.

Proof of Proposition 2

The only elements not established in the text are the price thresholds and the constants in Formula (7). For a current price level p, sufficiently high relative to p', that is for θ smaller than a critical threshold θ^*, it is optimal to establish Species P immediately; then, by (5), $F(p,p') = bp$ or $f(\theta) = b$. On the boundary $\theta = \theta^*$, the value-matching and smooth-pasting conditions (Dixit, 1993) imply $f(\theta^*) = b$ and $f_\theta(\theta^*) = 0$. Similarly, at levels p' sufficiently high relatively to p, that is for θ bigger than a critical threshold θ'^*, it is optimal to plant P' immediately; then $F(p,p') = b'p'$ or $f(\theta) = b'\theta$, and the value-matching and smooth-pasting conditions are respectively $f(\theta'^*) = b'\theta'^*$ and $f_\theta(\theta'^*) = b'$. Substituting (7) gives four equations that determine b_1, b_2, θ^* and θ'^*:

$$\begin{aligned} b_1 {\theta^*}^{\beta_1} + b_2 {\theta^*}^{\beta_2} &= b && \text{(25a)} \\ \beta_1 b_1 {\theta^*}^{\beta_1 - 1} + \beta_2 b_2 {\theta^*}^{\beta_2 - 1} &= 0 && \text{(25b)} \\ b_1 {\theta^{*'}}^{\beta_1} + b_2 {\theta^{*'}}^{\beta_2} &= b'\theta^{*'} && \text{(25c)} \\ \beta_1 b_1 {\theta^{*'}}^{\beta_1 - 1} + \beta_2 b_2 {\theta^{*'}}^{\beta_2 - 1} &= b' && \text{(25d)} \end{aligned}$$

Consider the first two equations $(25a)$ and $(25b)$; they imply:
$\beta_1 b_1 \theta^{*\beta_1} + \beta_1 b_2 \theta^{*\beta_2} = b\beta_1$
$\beta_1 b_1 \theta^{*\beta_1} + \beta_2 b_2 \theta^{*\beta_2} = 0$
so that $\left(\beta_1 - \beta_2\right) b_2 \theta^{*\beta_2} = b\beta_1$ or $\theta^* = \left(\frac{b\beta_1}{\beta_1 - \beta_2}\right)^{\beta_2^{-1}} b_2^{-\beta_2^{-1}}$.
The same first two equations also imply:
$\beta_2 b_1 \theta^{*\beta_1} + \beta_2 b_2 \theta^{*\beta_2} = b\beta_2$
$\beta_1 b_1 \theta^{*\beta_1} + \beta_2 b_2 \theta^{*\beta_2} = 0$
so that $\left(\beta_1 - \beta_2\right) b_1 \theta^{*\beta_1} = -b\beta_2$ or $\theta^* = \left(\frac{-b\beta_2}{\beta_1 - \beta_2}\right)^{\beta_1^{-1}} b_1^{-\beta_1^{-1}}$.
Eliminating θ^* gives

$$\left(\frac{b\beta_1}{\beta_1 - \beta_2}\right)^{\beta_1} b_1^{\beta_2} = \left(\frac{-b\beta_2}{\beta_1 - \beta_2}\right)^{\beta_2} b_2^{\beta_1} \tag{26}$$

Consider $(25c)$ and $(25d)$; they imply
$\left(\beta_1 - \beta_2\right) b_2 \theta^{*'\beta_2} = b' \theta^{*\prime} \left(\beta_1 - 1\right)$
$\left(\beta_1 - \beta_2\right) b_1 \theta^{*'\beta_1} = b' \theta^{*\prime} \left(1 - \beta_2\right)$
and by eliminating $\theta^{*'}$

$$\left(\frac{b'\left(\beta_1 - 1\right)}{\beta_1 - \beta_2}\right)^{\beta_1 - 1} b_1^{\beta_2 - 1} = \left(\frac{b'\left(1 - \beta_2\right)}{\beta_1 - \beta_2}\right)^{\beta_2 - 1} b_2^{\beta_1 - 1}. \tag{27}$$

Solving for b_1 using equations (26) and (27):
$\left(\frac{b\beta_1}{\beta_1-\beta_2}\right)^{\beta_1(\beta_1-1)} b_1^{\beta_2(\beta_1-1)} = \left(\frac{-b\beta_2}{\beta_1-\beta_2}\right)^{\beta_2(\beta_1-1)} b_2^{\beta_1(\beta_1-1)}$
$\left(\frac{b'(\beta_1-1)}{\beta_1-\beta_2}\right)^{\beta_1(\beta_1-1)} b_1^{\beta_1(\beta_2-1)} = \left(\frac{b'(1-\beta_2)}{\beta_1-\beta_2}\right)^{\beta_1(\beta_2-1)} b_2^{\beta_1(\beta_1-1)}$
$b_1^{\beta_1-\beta_2} = \left(\frac{-b\beta_2}{\beta_1-\beta_2}\right)^{\beta_2(\beta_1-1)} \left(\frac{b'(1-\beta_2)}{\beta_1-\beta_2}\right)^{-\beta_1(\beta_2-1)} \left(\frac{b\beta_1}{\beta_1-\beta_2}\right)^{-\beta_1(\beta_1-1)} \left(\frac{b'(\beta_1-1)}{\beta_1-\beta_2}\right)^{\beta_1(\beta_1-1)}$
$b_1^{\beta_1-\beta_2} = \left(\frac{1-\beta_2}{\beta_1-\beta_2}\right)^{\beta_1-\beta_2} \left(\frac{-\beta_2}{\beta_1-\beta_2}\right)^{\beta_2(\beta_1-1)} \left(\frac{1-\beta_2}{\beta_1-\beta_2}\right)^{-\beta_2(\beta_1-1)} \left(\frac{\beta_1}{\beta_1-1}\right)^{-\beta_1(\beta_1-1)} b^{-(\beta_1-1)(\beta_1-\beta_2)} b'^{\beta_1(\beta_1-\beta_2)}$
$b_1^{\beta_1-\beta_2} = \left(\frac{1-\beta_2}{\beta_1-\beta_2}\right)^{\beta_1-\beta_2} \left(\frac{\beta_2}{\beta_2-1}\right)^{\beta_2(\beta_1-1)} \left(\frac{\beta_1}{\beta_1-1}\right)^{-\beta_1(\beta_1-1)} b^{-(\beta_1-1)(\beta_1-\beta_2)} b'^{\beta_1(\beta_1-\beta_2)}$
$b_1 = \frac{1-\beta_2}{\beta_1-\beta_2} \left(\frac{\beta_1-1}{\beta_1}\right)^{\beta_1(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{-\beta_2(\beta_1-1)/(\beta_1-\beta_2)} \frac{b'^{\beta_1}}{b^{\beta_1-1}}$
Similarly for b_2,
$b_2 = \frac{\beta_1-1}{\beta_1-\beta_2} \left(\frac{\beta_2-1}{\beta_2}\right)^{\beta_2(1-\beta_2)/(\beta_1-\beta_2)} \left(\frac{\beta_1-1}{\beta_1}\right)^{\beta_1(\beta_2-1)/(\beta_1-\beta_2)} \frac{b'^{\beta_2}}{b^{\beta_2-1}}$.
Substituting for b_1 in $\theta^{*\beta_1} = \frac{-b\beta_2}{\beta_1-\beta_2} b_1^{-1}$, one has
$\theta^{*\beta_1} = \frac{-b\beta_2}{\beta_1-\beta_2} \frac{\beta_1-\beta_2}{1-\beta_2} \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{1-\beta_2}{-\beta_2}\right)^{\beta_2(\beta_1-1)/(\beta_1-\beta_2)} \frac{b^{\beta_1-1}}{b'^{\beta_1}}$
$\theta^{*\beta_1} = \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{1-\beta_2}{-\beta_2}\right)^{\beta_2(\beta_1-1)/(\beta_1-\beta_2)-1} \frac{b^{\beta_1}}{b'^{\beta_1}}$

$\theta^{*\beta_1} = \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{1-\beta_2}{-\beta_2}\right)^{\beta_1(\beta_2-1)/(\beta_1-\beta_2)} \frac{b^{\beta_1}}{b'^{\beta_1}}$

$\theta^{*} = \left(\frac{\beta_1-1}{\beta_1}\right)^{-(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{(\beta_2-1)/(\beta_1-\beta_2)} \frac{b}{b'}$

Similarly, substituting for b_2 in $(\beta_1-\beta_2)\, b_2\theta^{*'\beta_2} = b'\theta^{*'}(\beta_1-1)$, one has.

$\theta^{*'} = \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{\beta_2/(\beta_1-\beta_2)} \frac{b}{b'}$. To sum up:

$$b_1 = \frac{1-\beta_2}{\beta_1-\beta_2}\left(\frac{\beta_1-1}{\beta_1}\right)^{\beta_1(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{-\beta_2(\beta_1-1)/(\beta_1-\beta_2)} \frac{b'^{\beta_1}}{b^{\beta_1-1}} \tag{28a}$$

$$b_2 = \frac{\beta_1-1}{\beta_1-\beta_2}\left(\frac{\beta_2-1}{\beta_2}\right)^{\beta_2(1-\beta_2)/(\beta_1-\beta_2)} \left(\frac{\beta_1-1}{\beta_1}\right)^{\beta_1(\beta_2-1)/(\beta_1-\beta_2)} \frac{b'^{\beta_2}}{b^{\beta_2-1}} \tag{28b}$$

$$\theta^{*} = \left(\frac{\beta_1-1}{\beta_1}\right)^{-(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{(\beta_2-1)/(\beta_1-\beta_2)} \frac{b}{b'} \tag{28c}$$

$$\theta^{*'} = \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{\beta_2/(\beta_1-\beta_2)} \frac{b}{b'} \tag{28d}$$

Proof of Lemma 1

$F(p)$ may be written as $F(p) = \max\limits_{\{a_k\}} E\sum_{k=1}^{\infty} e^{-r\tau_k} p_{\tau_k} V(a_k)$ where current time is $t = 0$ and $\tau_k = \sum_{s=1}^{k} a_s$ is the sum of all harvest ages from the next harvest to the k^{th} harvest. Since p is a GBM, multiplying p by λ implies that p_s is multiplied by λ for any $s > t$. Then $F(\lambda p) = \max\limits_{\{a_k\}} E\sum_{k=1}^{\infty} e^{-r\tau_k}(\lambda p_{\tau_k})V(a_k)$

$= \lambda \max\limits_{\{a_k\}} E\sum_{k=1}^{\infty} e^{-r\tau_k} p_{\tau_k} V(a_k) = \lambda F(p)$

Consequently, (8) can be written as

$pF(1) = \max\limits_{s} E e^{-rs}\left[p_s V(s) + p_s F(1)\right]$ or $pF(1) = \max\limits_{s}\left[pe^{-\delta s}V(s) + pe^{-\mu s}F(1)\right]$

$= p\max\limits_{s}\left[e^{-\delta s}V(s) + e^{-\mu s}F(1)\right]$. Thus the maximization which defines the optimal harvest age is independent of the price.

Proof of Equation (17)

$G(p,p',a) = pg(\theta,a)$ must satisfy Bellman equation $EdG(p,p',a) = rG(p,p',a)da$.

As $dG = G_a da + G_p dp + \frac{1}{2}G_{pp}dp^2 + G_{p'}dp' + \frac{1}{2}G_{p'p'}dp'^2 + G_{p'p}dpdp' + o(da)$ where $G_a(p,p',a) = pg_a(\theta,a)$,

$G_p(p,p',a) = g(\theta,a) - \theta g_\theta(\theta,a)$,

$G_{pp}(p,p',a) = \frac{\theta^2}{p}g_{\theta\theta}(\theta,a)$,

$G_{p'}(p,p',a) = g_\theta(\theta,a)$,

$G_{p'p'}(a,p,p') = \frac{1}{p}g_{\theta\theta}(\theta,a)$, and

$G_{pp'}(p,p',a) = -\frac{\theta}{p}g_{\theta\theta}(\theta,a)$.

Then $EdG(p,p',a)$

$= pg_a da + \mu pg da + p\theta g_\theta(\mu'-\mu)da + \theta^2 g_{\theta\theta}\left(\frac{\sigma^2}{2} - \rho\sigma\sigma' + \frac{\sigma'^2}{2}\right)da + o(da)$,

so that $\frac{\overline{\sigma}^2}{2}\theta^2 g_{\theta\theta} + \overline{\mu}\theta g_\theta - \delta g + g_a = 0$ where $\overline{\sigma}^2 = \sigma^2 - 2\rho\sigma\sigma' + \sigma'^2$ and $\overline{\mu} = \mu' - \mu$.

Proof of Proposition 4

The continuity of $a^+(\theta)$ follows from the continuity of the functions g, f, and V that define it in expressions (18a), (18a), and (18a). This argument also applies at θ^* because functions g and f are continuous at θ^* despite the change of definitions that they experience at that point according to (13c). The continuity of $a^-(\theta)$ follows from the continuity of the functions that define it in (16).

When θ tends to zero, the problem collapses to the single species case analyzed above, for which the optimum harvest age is a_f given by (9) i.e. $a^+(0) = a_f$. When θ is sufficiently close to zero, Equation (17) implies $-\delta g + g_a \simeq 0$ or, using (18a) and (18c), $\delta f(\theta) \simeq V_a(a^+(\theta)) - \delta V(a^+(\theta))$ where $f(\theta)$ is non decreasing in θ while $V_a(a) - \delta V(a)$ strictly decreases in a. It follows that $a^+(\theta)$ is non increasing at low values of θ.

Let us show that $a^+(\theta)$ is strictly decreasing on $\left[\theta^*, \overline{\theta}\right]$. Assume that there exists a point $A(\theta, a)$ on the boundary where $a^+(\theta)$ is non decreasing while $\theta > \theta^*$. At $A(\theta, a)$ the investor is indifferent between harvesting now and establishing Species P' (the left-hand side of (29)), or waiting and harvesting later (the right-hand side of (29))[28]:

$$V(a)+\max_t\left[e^{-\delta' t}\theta V'(t) + e^{-rt}E\left\{\frac{p_t}{p}f(\theta_t)\right\}\right] = \max_s\left[e^{-\delta s}V(a+s) + e^{-rs}E\left\{\frac{p_s}{p}f(\theta_s)\right\}\right] \tag{29}$$

where E denotes expectation, conditional on $\frac{p_s}{p}$ and θ_s having initial values of 1 and θ respectively.

Intuitively, if the upper boundary $a^+(\theta)$ is increasing at point $A(\theta, a)$ then point $B'(\theta + d\theta, a)$, for $d\theta$ sufficiently small and negative, is above the upper boundary implying that it is preferable to harvest immediately at point $B'(\theta + d\theta, a)$. This is impossible as the trees to be harvested at A and B' have the same age but Species P has more value relative to Species P' at B' than at A, which calls for a higher harvest age in the maximization on the right-hand-side of (29). More formally, let $\varphi(s, y)$ denote $E\left\{\frac{p_s}{p}f(\theta_s)\right\}$ where $y_s = \frac{p_s}{p}f(\theta_s)$ follows a generalized Brownian motion with $y = f(\theta)$.[29] Let a small period of time $da > 0$ elapse, over which θ becomes $\theta + d\theta$ and y becomes $y + dy$, where dy has the same sign as $d\theta$. Take the case $d\theta < 0$ and consider the point $B(\theta + d\theta, a + da)$, to the upper left of A. Since $a^+(\theta)$ is assumed to be non decreasing, point B lies inside the immediate-harvest region. It follows that at B:

$$V(a+da)+\max_t\left[e^{-\delta' t}(\theta + d\theta)V'(t) + e^{-rt}\varphi(t, y+dy)\right] > \max_s\left[e^{-\delta s}V(a+da+s) + e^{-rs}\varphi(s, y+dy)\right] \tag{30}$$

where $\varphi(s, y+dy) = \varphi(s, y) + \varphi_y(s, y)Edy + \frac{1}{2}\varphi_{yy}(s, y)Edy^2$.

The Kolmogorov backward differential equation associated with $\varphi(s, y)$ (see for instance Karlin et al. (1981), page 214) is $\varphi_y(s, y)Edy + \frac{1}{2}\varphi_{yy}(s, y)Edy^2 - \varphi_s(s, y)\,da = 0$; then $\varphi(s, y+dy) = \varphi(s, y) + \varphi_s(s, y)\,da$ or $\varphi(s, y+dy) = \varphi(s, y) + O(da)$ where $O(da)$ denotes terms of magnitude smaller than or equal to da that are negligible relative to $d\theta$. Substituting $V(a) + O(da)$ for $V(a+da)$ and $V(a+s) + O(da)$ for $V(a+da+s)$ in (30)

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[28] Both sides of the equation are normalized by p.

[29] Recall the convention introduced earlier about time subscripts: θ, p, y refer to current values, while θ_s, p_s, y_s refer to the values of the same variables after a period s has elapsed; similarly, E is conditional on current information p, p', or θ.

and neglecting terms involving $O(da)$ under the assumption that $da \downarrow 0$, (30) becomes $V\,(a)+\max\limits_{t}\left[e^{-\delta' t}(\theta+d\theta)V'(t)+e^{-rt}\varphi\,(t,y)\right] > \max\limits_{s}\left[e^{-\delta s}V(a+s)+e^{-rs}\varphi(s,y)\right]$ which contradicts Equation (29) as the later implies $V\,(a)+\max\limits_{t}\left[e^{-\delta' t}(\theta+d\theta)V'(t)+e^{-rt}\varphi\,(t,y)\right] < \max\limits_{s}\left[e^{-\delta s}V(a+s)+e^{-rs}\varphi(s,y)\right]$ for $d\theta<0$. Therefore $a^{+}\,(\theta)$ is strictly decreasing on $\left]\theta^{*},\overline{\theta}\right]$.[30]

Similarly, let us show that the lower boundary $a^{-}\,(\theta)$ is strictly increasing on $\left[\overline{\theta}',\ \theta^{*}\right]$. At a point $C(\theta,a)$ on $a^{-}\,(\theta)$, with $a<\underline{a}$ and $\overline{\theta}'<\theta<\ \theta^{*}$, assume that $a^{-}\,(\theta)$ is non increasing. The investor is indifferent between harvesting immediately thus earning the LEV (the harvest is worth zero as $a<\underline{a}$) given by the left-hand term of the following equation, or harvesting later at age $a+s>\underline{a}$, in which case she earns the right-hand of the same equation:

$$\max_{t}\left\{e^{-\delta' t}\theta V'(t)+e^{-rt}E\frac{p_t}{p}f\,(\theta_t)\right\}=\max_{s}\left\{e^{-\delta s}V(a+s)+e^{-rs}E\left[\frac{p_s}{p}f\,(\theta_s)\right]\right\} \tag{31}$$

After a small time interval $da>0$, θ becomes $\theta+d\theta$. Assuming that $d\theta>0$ and $a^{-}\,(\theta)$ is non increasing, the pair $C(\theta,a)$ moves to $D(\theta+d\theta,a+da)$ in the continuation region. Proceeding as in the previous proof, we obtain $\max\limits_{t}\left[e^{-\delta' t}\,(\theta+d\theta)\,V'(t)+e^{-\delta t}E\frac{p_t}{p}f(\theta_t)\right] \leq \max\limits_{s}\left[e^{-\delta s}V(a+s)+e^{-\delta s}E\frac{p_s}{p}f(\theta_s)\right]$ which cannot hold along with Equation (31) as $d\theta>0$. This implies that the boundary $a^{-}\,(\theta)$ is strictly increasing on $\left[\theta^{*},\overline{\theta}\right]$.

At $(\theta^{*},0)$ it is equivalent to establish Species P or Species P'; thus $(\theta^{*},0)$ belongs to $a^{-}\,(\theta)$. Also, by definition, $a^{-}\,(\theta)\leq\underline{a}$; in particular $a^{-}\left(\overline{\theta}\right)\leq\underline{a}$. Let us show by contradiction that $a^{+}\left(\overline{\theta}\right)=\underline{a}$. Suppose that $a^{+}\left(\overline{\theta}\right)=\underline{a}+\varepsilon$, $\varepsilon>0$. Then $V(\underline{a}+\varepsilon)+f(\bar{\theta})>V(\underline{a})+f(\bar{\theta})\geq V(a^{-}\left(\overline{\theta}\right))+f(\bar{\theta})$ where the left-hand side applies the definition (15) of the upper boundary while the right-hand side results from the condition that $a^{-}\left(\overline{\theta}\right)\leq\underline{a}$. But then, for some pair $\left(\varepsilon'',\bar{\theta}''\right)$ such that $0<\varepsilon''<\varepsilon$ and $\theta''>\bar{\theta}$, it is also true that $V(\underline{a}+\varepsilon'')+f(\bar{\theta}'')>V(a^{-}\left(\overline{\theta}\right))+f(\bar{\theta}'')$ so that $\bar{\theta}$ is not the maximum value of θ at which the continuation region exists, contradicting its definition. This contradiction can be avoided only if $\varepsilon=0$, implying that $a^{+}\left(\overline{\theta}\right)=\underline{a}$. Now suppose $a^{-}\left(\overline{\theta}\right)<\underline{a}$; then by $(13a)$ $g\left(\overline{\theta},\underline{a}\right)>V\left(a^{-}\left(\overline{\theta}\right)\right)+f(\bar{\theta})=f(\bar{\theta})$. However, $g\left(\overline{\theta},a^{-}\left(\overline{\theta}\right)\right)=f(\bar{\theta})$ by (16) since $V\,(a)=0\ \forall\ a<\underline{a}$, a contradiction. Hence $a^{-}\left(\overline{\theta}\right)=a$. We conclude that the upper and lower boundaries meet at $(\bar{\theta},\underline{a})$.

Let us show now that $a^{+}(\theta)$ is strictly increasing on $[\overline{\theta}',\theta^{*}]$. As $a^{+}(\theta)$ is strictly decreasing on $[\theta^{*},\overline{\theta}]$ then, by symmetry (see Figure 1), $a'^{+}(\theta)$ is strictly increasing on $[\overline{\theta}',\theta^{*}]$. On this interval, $a^{+}(\theta)$ and $a'^{+}(\theta)$ are linked by Equation (23) whose left side $\theta\left[\delta' V'\left(a'^{+}\left(\theta\right)\right)-V_a'\left(a'^{+}\left(\theta\right)\right)\right]$ is strictly increasing in θ as $\delta' V'\left(a'\right)-V_a'\left(a'\right)$ is strictly increasing in a' and $a'^{+}\,(\theta)$ is strictly increasing in θ. The right side $\delta V\left(a^{+}\left(\theta\right)\right)-$

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[30] The solution of the maximization problems on both sides of the inequality are affected by the expansion of the objective functions. However the derivation is valid whether or not the optimum timing of the harvest is forced to be maintained while the functions are expanded. We avoid introducing further notation by implicitely ignoring this issue as it has no bearing on the result.

$V_a\left(a^+(\theta)\right)$ cannot be strictly increasing unless so is $a^+(\theta)$. We conclude that $a^+(\theta)$ is strictly increasing on $[\overline{\theta}', \theta^*]$.

APPENDIX B: THE COMPUTATIONAL ALGORITHM

The partial differential equation governing $g(\theta, a)$ can be simplified by performing the change of variable $x = \log(\theta)$. Let $h(x, a) = g(\theta, a)$ and $l(x) = f(\theta)$, then the partial differential equation governing $h(x, a)$ is $\frac{\overline{\sigma}^2}{2} h_{xx} + \left(\overline{\mu} - \frac{\overline{\sigma}^2}{2}\right) h_x - \delta h + h_a = 0$.

Call $h(x, a)$ the Transformed Forest Value (TFV) function. To compute the TFV function, it is helpful to specify the corresponding optimal stopping problem as a linear complementarity one (Zvan et al., 1998). Let $\mathcal{L}$ be the linear operator defined as

$$\mathcal{L}.h = \frac{\overline{\sigma}^2}{2} h_{xx} + \left(\overline{\mu} - \frac{\overline{\sigma}^2}{2}\right) h_x - \delta h + h_a.$$

Then, the linear complementarity problem is

$$\begin{aligned} \mathcal{L}.h(x,a) &\geq 0, & (32a)\\ h(x,a) - (l(x) + V(a)) &\geq 0, & (32b)\\ \mathcal{L}.h(x,a)\left[h(x,a) - (l(x) + V(a))\right] &= 0, & (32c) \end{aligned}$$

Note that this formulation does not imply any explicit use of the free harvest boundary $a(\theta)$. It shows that the TFV function can be considered as the value of an American option with expiry date equal to infinity, an underlying asset which is a geometric diffusion process with drift $\overline{\mu}$ and volatility $\overline{\sigma}^2$, and a discount factor equal to δ. Indeed, in the continuation region where it is optimal to continue holding the option to harvest, the required return δh is equal to the actual return or equivalently $\mathcal{L}.h(x, a) = 0$ and the option value is higher than the payoff, that is $h(x, a) - (l(x) + V(a)) > 0$. Consequently, it is not yet optimal to exercise. In the stopping region, it is no longer optimal to continue holding the option to harvest but it is optimal to harvest immediately because the required return δh is less than the actual return or equivalently $\mathcal{L}.h(x, a) > 0$ and the option value must equal the payoff that is $h(x, a) - (l(x) + V(a)) = 0$. The free harvest boundary is just where the decision maker is indifferent between harvesting immediately or continuing to hold this option, that is when $\mathcal{L}.h(x, a) = 0$ and $h(x, a) - (l(x) + V(a)) = 0$.

When valuing an American option, the use of the complementarity formulation is straightforward. The option value is then computed numerically by performing a discretization of the linear complementarity problem. The main difference between valuing an American option and the current problem is that the payoff in the former case is a known function of the underlying asset whereas it should be endogenously determined in the latter case as the payoff (in units of the numeraire) is the sum of the timber value and the land value $l(x) = h(x, 0)$. For this reason, valuing the TFV function $h(x, a)$ as a complementarity problem as specified above is not possible. Instead, we need to consider simultaneously the companion problem of valuing the TFV function for Species P' $g'(\theta, a) \equiv h'(x, a)$:

$$
\begin{aligned}
\mathcal{L}.h'(x,a) &\geq 0, & \text{(33a)}\\
h'(x,a) - (l(x) + e^x V(a)) &\geq 0, & \text{(33b)}\\
\mathcal{L}.h'(x,a)\,[h'(x,a) - (l(x) + e^x V(a))] &= 0, & \text{(33c)}
\end{aligned}
$$

$$
l(x) = h(x,0) = h'(x,0). \tag{34}
$$

Equations $(32a)-(33c)$ represent the complementarity problem respectively for stand P and stand P' whereas equation (34) says that each TFV function at age zero must, by Proposition 3, equal the reduced land value function or its transformed version the TLV function $l(x)$. As specified by equations $(32a)-(33c)$, and equation (34), the problem of valuing both stands simultaneously can then be solved numerically.

The TFV functions $h(x,a)$ and $h'(x,a)$ and the TLV function $l\,(x)$ are computed iteratively as specified by the following pseudo-code. First, (i) assume that after the $n-1^{th}$ iteration, the TLV function is $l^{(n)}$; then (ii) compute the TFV functions $h^{(n)}$ and $h'^{(n)}$ as solutions to $(32a)-(32c)$ and $(33a)-(33c)$ respectively; (iii) deduce the new TLV function as $l^{(n+1)}(x) = \max(h^{(n)}(x,0), h'^{(n)}(x,0))$; (iv) continue iteratively until convergence is reached when $h(x,0)$ and $h'(x,0)$ are approximately equal. Once the land value is determined, the harvest age boundary is localized using the value-matching $(18a)$ and smooth-pasting $(18b)$ conditions.

The initial value ascribed to the TLV function can be arbitrarily chosen. We find that the algorithm converges when the TLV function is initially ascribed a positive value. However, the convergence is generally faster when the initial value is $\max(c, c'\theta)$, the minimal reduced land value when both tree species P and P' are available.

The numerical discretization to compute $h^{(n)}$ and $h'^{(n)}$ at step (ii) is based on a fully implicit finite difference method. Compared with a fully explicit finite difference method; the implicit method is unconditionally stable and more robust (Brennan and Schwartz 1978). Other numerical methods to solve option valuation problems are discussed in Wilmott et al. (1993). We use the penalty method (Zvan et al., 1998) to solve the linear complementarity problem as in Insley (2002).

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