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André Lemelin

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Endogenous savings rate with forward-looking households in a recursive dynamic CGE model: application to South Africa*

André Lemelin†

Résumé/abstract

Dans la plupart des modèles d'équilibre général calculable dynamiques séquentiels, le taux d'épargne est constant et exogène. Les modèles intertemporels, eux, sont résolus simultanément pour toutes les périodes et les agents pratiquent l'optimisation intertemporelle. Mais la cohérence théorique de l'optimisation intertemporelle n'est atteinte qu'au prix de modèles moins détaillés, à cause de limites sur le volume des calculs. C'est pourquoi, quand le détail des résultats est important, on peut préférer utiliser un modèle dynamique séquentiel. Cet article présente un modèle d'équilibre général calculable dynamique séquentiel dans lequel les ménages déterminent leur taux d'épargne par l'optimisation intertemporelle, en résolvant une forme simplifiée de leur problème intertemporel. C'est ce que nous appelons des « anticipations rationnelles tronquées » (TRE). Dans ce cadre, les ménages ont des anticipations rationnelles pour la période courante et la suivante. Le modèle est donc résolu simultanément pour deux périodes à la fois, la courante τ et la suivante. Les anticipations rationnelles des ménages pour la période $\tau+1$ sont données par la solution du modèle. Pour les périodes subséquentes, les anticipations sont formées par extrapolation à partir des valeurs de τ et $\tau+1$, en supposant un taux de changement constant. L'approche TRE est implantée dans une version modifiée du modèle PEP-1-t de Decaluwé *et al.* (2013), au moyen d'une matrice de comptabilité sociale (MCS) de l'Afrique du Sud pour 2005 due à Davies and Thurlow (2011). Différentes simulations sont menées, avec deux variantes de la MCS, l'originale et une version modifiée avec un taux élevé d'épargne des ménages. Les résultats sont comparés avec ceux d'un modèle avec anticipations statiques et optimisation intertemporelle, et avec ceux d'un modèle à taux d'épargne fixe. La principale différence observée est que dans les deux premiers modèles, le taux d'épargne des ménages n'est pas constant, même dans le scénario de référence. De plus, il réagit aux variations du taux de rendement des actifs. Par contre, une réduction exogène du stock de richesse des ménage a très peu d'impact.

Mots clés : Modèles d'équilibre général calculable, modèles dynamiques séquentiels, optimisation intertemporelle, épargne des ménages .

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† Centre INRS-UCS, Université du Québec, Montréal andre_lemelin@ucs.inrs.ca

In the vast majority of recursive dynamic CGE models, the savings rate is constant and exogenous. Intertemporal CGE models, by contrast, are solved simultaneously for all periods, and agents optimize intertemporally. But the theoretical consistency of intertemporal optimization is achieved only at the cost of more aggregated, less detailed models, due to computational limitations. In some applications, therefore, recursive dynamics should be preferred to intertemporal dynamics for practical reasons of computability. This paper presents a recursive dynamic CGE model in which households determine their savings rate from intertemporal optimization, by solving a simplified form of their intertemporal problem. This approach we call « truncated rational expectations » (TRE). In the TRE framework, households have rational expectations for the current period and the following one. Accordingly, the model is solved simultaneously for two periods at a time, the current period τ and the following one. Household (rational) expectations for period τ +1 are given by the model solution. For subsequent periods, household expectations are formed by extrapolating from τ and τ +1 solution values, assuming a constant rate of change. The TRE framework is implemented in a modified version of the Decaluwé et al. (2013) PEP-1-t model, and applied to South Africa, using a 2005 SAM by Davies and Thurlow (2011). Several simulations are run, with two variants of the 2005 SAM, the original one and a modified one with a high initial household savings rate. The results are compared with those of a static expectations model with intertemporal optimization, and of a fixed-savings rate model. The main difference is that in the first two models, the household savings rate is not constant, even in the BAU scenario. It is also responsive to changes in the rate of return on assets. On the other hand, an exogenous reduction in household wealth has very little effect.

Key words: *Computable general equilibrium models, recursive dynamic models, intertemporal optimization, household savings*

Codes JEL *:* C68, D1, D58, D91

Contents

Introduction

Dynamic computable general equilibrium (CGE) models may be classified into recursive and intertemporal. Recursive dynamic CGE models are solved as a succession of static models, one period at a time, each period inheriting some exogenous variables (most notably the stock of capital) from the previous one[1]. In such models, households and other economic agents optimize their choices without knowing the future, and, most of the time, ignoring it (in recursive dynamic CGE models, expectations, even adaptive expectations, generally play no part in agents' behavior). Intertemporal CGE models, by contrast, are solved simultaneously for all periods, and agents optimize intertemporally. Intertemporal models may be deterministic or stochastic. In deterministic intertemporal models, agents are usually gifted with perfect foresight. In dynamic stochastic general equilibrium (DSGE) models, agents are faced with uncertainty, and they optimize the mathematical expectation of future outcomes[2].

We concur with Babiker *et al.* (2009) that neither the recursive nor the intertemporal approach dominates the other. Comparing two versions of the MIT EPPA model, one recursive, the other intertemporal, they point out that « the forward-looking model also had to be simplified in some regards to make it computationally feasible », and draw the following conclusion:

> « Economists typically consider forward-looking models a significant advance over the recursive structure because in reality agents expectations about the future affect current behavior. However, for policy purposes the tradeoff of less structural detail and the assumption of perfect foresight over all time (as opposed to uncertain expectations of what might happen in the future) leaves open the question of which formulation gives more realistic answers. Some problems simply demand the forward-looking structure to get at basic issues, while for others the recursive structure may be more realistic. We thus see these two versions as complementary. » (p.1342)

So the theoretical consistency of intertemporal optimization is achieved only at the cost of more aggregated, less detailed models, due to computational limitations. In some applications, therefore, recursive dynamics should be preferred to intertemporal dynamics for practical reasons of computability.

Moreover, the rational expectations paradigm which underlies intertemporal models has been criticized on the grounds that « DSGE models (and more generally of macroeconomic models based on rational expectations) [...] assume extraordinary cognitive capabilities of individual agents » (de Grauwe, 2010;

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1 The reader must be warned that usage of the word « recursive » is not fully stabilized, witness the Wikipedia article on « Recursive economics », which refers to what we call here « intertemporal ».

2 Uncertainty in these models often takes the form of « technological shocks » on multi-factor productivity, with known probability distribution. Some DSGE models are solved each period for all time up to the horizon, on the basis of past and current outcomes.

also see de Grauwe, 2012) . The same author then goes on to argue that « we need models that take into account the limited cognitive abilities of agents ».

This argument, together with the practical limitations of intertemporal models, has prompted us to follow H.A. Simon (1978, 1982), and make the hypothesis that, under the bounded rationality principle, households solve a simplified form of their intertemporal problem. This approach, which we call « boundedly rational expectations », is what we develop in this paper.[3]

More specifically, we apply the bounded rationality principle to household decisions regarding the allocation of their income between savings and consumption. In addition to the motivations expressed above, an objective of this project is to develop an operational CGE modelling approach to endogenize the household savings rate in a recursive dynamic model. Specifically, we want the household savings rate to be responsive to changes in the rate of return on assets. Furthermore, it should reflect the adjustment behavior of households facing economic shocks that change the value of their capital endowment. And in the true spirit of CGE modeling, we want our model to rest upon a sound theoretical basis, which is why a formulation with forward-looking households is so attractive.

In the vast majority of CGE models, the savings rate is constant and exogenous. In some models, such as the IFPRI standard model (Löfgren *et al.*, 2002), the household savings rate may be endogenously determined so as to accommodate some exogenous amount of investment expenditures (SI-1 closure). In other models, savings are treated as a proxy for future consumption, and are jointly determined with household current consumption expenditure[4].

An earlier paper (Lemelin 2012) put forth a theoretical model of the representative household that meets these objectives. In the model, household consumption expenditures and savings are determined under intertemporal optimization. Two variants of the model were examined. In the first, households have static expectations: they expect consumer prices, their non-investment income and the rate of return on savings to remain at their current values indefinitely[5]. In the second variant, households make their decisions on the basis of boundedly-rational, or truncated rational expectations: households are assumed to have near-perfect foresight one period ahead and to extrapolate changes between the current and the next period into the future to form their expectations.

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3 Another approach to mitigate the stringency of the rational expectations hypothesis, applied in several DSGE models, is to consider two classes of households, one of which has rational expectations and acts accordingly, while the other (often called « constrained ») follows a more mechanical behavior.

4 An early example is the extended linear expenditure system (ELES) (Lluch, 1973; Howe, 1775). Also, see Part 2 in Lemelin and Decaluwé (2007). Similar approaches are found in the GTAP model and the MIT EPPA model.

5 Static expectations are also called « myopic expectations »: Evans and Honkapohja (2001).

Certainly, the proposed approach, especially truncated rational expectations, poses implementation challenges. It was first applied, with success, in a « toy model » in which reduced size minimized the computational burden so that the main programming issues could be resolved.[6] But it remained to be shown that the approach is applicable to a « real life » situation. This is what this paper demonstrates, by applying the truncated rational expectations model to South Africa, 2005.

The first part of the paper briefly reviews the underlying theory. The second part describes how the theoretical model is implemented in the PEP-1-t standard recursive dynamic CGE model (Decaluwé *et al*., 2013). The third part presents an application to South Africa, 2005, and discusses some preliminary results. A brief conclusion assesses what remains to be done for the model to become fully operational and applicable in a general context.

1. Theoretical (re)formulation of the household intertemporal problem

Terminology and notation

In this essay, the expression « current assets » designates assets owned by the household at the *beginning* of the current period. The expression « terminal wealth » or « residual wealth » means assets to be left at the end of the final period of the household's planning time-span, occasionally called « target assets ». In the models examined, there are two sources of income, investment income and non-investment income; when the word « income » is not qualified, it means non-investment income. The expression « dynamic budget constraint » designates the household's single-period constraint; the intertemporal budget constraint, also called the lifetime budget constraint, covers the entire household planning time-span.

Theoretical equations are numbered [ttt001] to [ttt047]. PEP model equations are designated by an « M » followed by a three-digit number. Other equations are numbered [iii 001] to [iii 023].

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6 Results were presented at the 53e annual meeting of the Société canadienne de science économique, Québec, May 15-17, 2013 under the title « L'épargne des ménages dans un MEGC dynamique séquentiel avec optimisation intertemporelle et anticipations rationnelles tronquées », and later at the 47th annual meeting of the Canadian Economics Association, Montréal, May 30-June 2, 2013: « Household savings in a recursive dynamic CGE model with intertemporal optimization and truncated rational expectations ».

1.1 Intertemporal problem with non-depreciating asset

Start with the household intertemporal problem in its standard theoretical form

$$\max_{\{c_t\}_{t=\tau}^{T}} U_\tau = \sum_{t=\tau}^{T} \beta^{t-\tau} u(c_t) \qquad \text{[ttt001]}$$

subject to the dynamic budget constraint

$$a_{t+1} = (1+r_t)(a_t + y_t - p_t c_t),\ t = \tau,\ldots,T \qquad \text{[ttt002]}$$

and the transversality (no Ponzi game) condition

$$a_{T+1} = \overline{a_{T+1}} \qquad \text{[vvv002]}$$

where

c_t is the volume of consumption in period t

β is the subjective discount factor: $\beta = 1/(1+\psi)$

ψ is the psychological discount rate, or time-preference, or rate of impatience

$u(c_t)$ is the single-period utility function

a_t is the nominal value of the household's assets (or wealth, or capital endowment) at the beginning of period t

p_t is the price index of consumption in period t

y_t is the household's nominal non-investment income in period t

r_t is the nominal rate of interest in period t

and where the single-period utility function is specified as the CRRA[7] utility function

$$u(c_t) = \frac{c_t^{1-\varsigma}}{1-\varsigma} \qquad \text{[ttt003]}$$

with intertemporal elasticity of substitution (IES) $\sigma = 1/\varsigma$. Dynamic budget constraint [ttt002] is equivalent to the intertemporal budget constraint

$$p_\tau c_\tau + \sum_{t=\tau+1}^{T}\left[\prod_{\theta=\tau}^{t-1}\left(\frac{1}{1+r_\theta}\right)\right] p_t c_t + \prod_{\theta=\tau}^{T}\left(\frac{1}{1+r_\theta}\right) a_{T+1} = a_\tau + y_\tau + \sum_{t=\tau+1}^{T}\left[\prod_{\theta=\tau}^{t-1}\left(\frac{1}{1+r_\theta}\right)\right] y_t \qquad \text{[ttt004]}$$

---

7 Constant Relative Risk Aversion. Actually, risk is absent from this model, so « risk aversion » merely characterizes the shape of the utility function.

where τ is the current period, and T is the final planning period currently considered by households. Maximizing [ttt001] subject to [ttt004] yields the first-order conditions from which is derived the Euler equation

$$(1+r_\tau)\beta\, u'(c_{\theta+1}) = u'(c_\theta) \qquad \text{[ttt005]}$$

1.2 Intertemporal problem with depreciating asset and classic accumulation rule

The dynamic budget constraint in [ttt002], however, assumes that household assets do not depreciate, and that current savings generate interest income. To take explicit account of depreciation and of the delay in income generation from assets, let us reformulate the problem in terms of capital [8] within a simple economywide model. Let the (classic) capital accumulation rule be

$$k_{t+1} = (1-\delta)k_t + I_t / p_{k,t} \qquad \text{[ttt006]}$$

where

k_t is the stock of capital in period t

δ is the rate of depreciation

I_t is the amount of investment expenditures in period t

$p_{k,t}$ is the price of the investment good in period t

Under the classic capital accumulation rule, new capital created by current investment becomes productive only in the following period. This specification is at variance with the assumption, made in the theoretical model of section 1.1, that current savings generate interest income.

Next, let z_t be income, and the budget constraint is

$$z_t = p_{c,t}c_t + I_t \qquad \text{[ttt007]}$$

where

c_t is consumption in period t

$p_{c,t}$ is the price of the consumption good in period t

So we have

$$k_{t+1} = (1-\delta)k_t + \frac{z_t - p_{c,t}c_t}{p_{k,t}} \qquad \text{[ttt008]}$$

---

8 Our starting point in elaborating what follows is the exposition in Wälde (2011).

For the sake of this theoretical model, we assume perfect factor mobility and constant-returns-to-scale production functions[9]. Perfect mobility implies that factor prices are equal between the consumer goods producing industry and the investment goods producing industry. Constant returns to scale imply

$$c_t = \frac{\partial c_t}{\partial k_{c,t}} k_{c,t} + \frac{\partial c_t}{\partial l_{c,t}} l_{c,t} = \frac{w_{k,t} k_{c,t} + w_{l,t} l_{c,t}}{p_{c,t}} \quad \text{[ttt009]}$$

$$\frac{I_t}{p_{k,t}} = \frac{\partial \left(I_t / p_{k,t} \right)}{\partial k_{k,t}} k_{k,t} + \frac{\partial \left(I_t / p_{k,t} \right)}{\partial l_{k,t}} l_{k,t} = \frac{w_{k,t} k_{k,t} + w_{l,t} l_{k,t}}{p_{k,t}} \quad \text{[ttt010]}$$

where

$k_{i,t}$ is the quantity of capital used in industry *i* in period *t*

$l_{i,t}$ is the quantity of labor used in industry *i* in period *t*

$w_{k,t}$ is the rental rate of capital in period *t*

$w_{l,t}$ is the wage rate in period *t*

Factor market equilibrium requires

$$l_t = l_{c,t} + l_{k,t} \quad \text{[ttt011]}$$

$$k_t = k_{c,t} + k_{k,t} \quad \text{[ttt012]}$$

Let

$$y_t = w_{l,t} l_t \quad \text{[ttt013]}$$

$$w_{k,t} = \rho_t p_{k,t} \quad \text{[ttt014]}$$

Then, substituting [ttt009], [ttt010], [ttt011], [ttt012], [ttt013], and [ttt014] into [ttt007]

$$z_t = p_{c,t} c_t + I_t = \rho_t p_{k,t} k_t + y_t \quad \text{[ttt015]}$$

Next, substitute [ttt015] into [ttt008]:

$$k_{t+1} = (1-\delta) k_t + \frac{\rho_t p_{k,t} k_t + y_t - p_{c,t} c_t}{p_{k,t}} \quad \text{[ttt016]}$$

$$p_{k,t} k_{t+1} = (1-\delta) p_{k,t} k_t + \rho_t p_{k,t} k_t + y_t - p_{c,t} c_t \quad \text{[ttt017]}$$

$$p_{k,t} k_{t+1} = \left(1 + \rho_t - \delta\right) p_{k,t} k_t + y_t - p_{c,t} c_t \quad \text{[ttt018]}$$

9 In the applied model presented later, we no longer assume perfect factor mobility.

Compared with equation [ttt002],

$$a_{t+1} = (1+r_t)(a_t + y_t - p_t c_t),\ t = \tau,\ldots,T \qquad \text{[ttt002]}$$

there are two differences. The first is that the rate of return r_t is replaced by $\rho_t - \delta$ to take account of depreciation. The second difference is that the surplus of labor-income over consumption does not yield any return under an accumulation rule according to which new capital created by investment becomes operational with a one-period delay. However, contrary to first impression, $\rho_t - \delta$ is not the correct discount rate to compute present values in the intertemporal problem, as we shall see below.

Finally, we define current savings from the single-period budget constraint [ttt018] as

$$s_t = p_{k,t}(k_{t+1} - k_t) = (\rho_t - \delta) p_{k,t} k_t + y_t - p_{c,t} c_t \qquad \text{[ttt019]}$$

In view of capital accumulation rule [ttt006], it is clear that equation [ttt019] defines ***net*** savings. To verify, rewrite [ttt006] as

$$p_{k,t} k_{t+1} = (1-\delta) p_{k,t} k_t + I_t \qquad \text{[ttt020]}$$

$$p_{k,t}(k_{t+1} - k_t) = I_t - \delta\, p_{k,t} k_t \qquad \text{[ttt021]}$$

Savings as defined in [ttt019] are indeed ***net*** savings, equal to net investments.

1.3 Tobin's Q and the discount rate

Suppose a capitalist purchases one unit of capital at the beginning of period t for a price of $p_{k,t}$. Since there is no vintage distinction within the stock of capital, acquiring one unit of existing capital or making an investment to create one unit of new capital must be equivalent. Therefore, capital acquired in period t yields income only in $t+1$, whether it is new or pre-existing capital[10]. So the scenario goes as follows.

A unit of capital acquired in period t for a price of $p_{k,t}$ will generate no income in period t, and an income of $w_{k,t+1}$ in period $t+1$; the $(1-\delta)$ fraction of capital remaining after depreciation may be re-sold in the same period $t+1$ for an amount of $(1-\delta)\, p_{k,t+1}$ to someone who will receive no income in $t+1$, and $(1-\delta)\, w_{k,t+2}$ in $t+2$, etc. The present value of the capitalist's income must be equal to the cost

[10] Pre-existing capital does generate income in the current period, but that income goes to the agent who owned the capital at the beginning of the current period.

of the investment:

$$p_{k,t} = \frac{w_{k,t+1}}{1+r_t} + (1-\delta)\frac{p_{k,t+1}}{1+r_t} \qquad \text{[ttt022]}$$

where r_t is the forward-looking discount rate. We have

$$(1+r_t)p_{k,t} = w_{k,t+1} + (1-\delta)p_{k,t+1} \qquad \text{[ttt023]}$$

$$(1+r_t)\frac{p_{k,t}}{p_{k,t+1}} = \frac{w_{k,t+1}}{p_{k,t+1}} + (1-\delta) \qquad \text{[ttt024]}$$

The ratio of the rental rate of capital on its price (replacement cost),

$$\rho_t = \frac{w_{k,t}}{p_{k,t}} \qquad \text{[ttt025]}$$

is the (gross) rate of return on capital. Also, define the growth fa ctor of the pr ice of investment goods $g_{pk,t}$, and the corresponding rate of inflation $\pi_{pk,t}$:

$$1+\pi_{pk,t} = g_{pk,t} = \left(\frac{p_{k,t+1}}{p_{k,t}}\right) \qquad \text{[ttt026]}$$

Then

$$\frac{(1+r_t)}{g_{pk,t}} = \rho_{t+1} + (1-\delta) \qquad \text{[ttt027]}$$

$$(1+r_t) = (1+\rho_{t+1}-\delta)g_{pk,t} = (1+\rho_{t+1}-\delta)(1+\pi_{pk,t}) \qquad \text{[ttt028]}$$

Equation [ttt028] defines the forward-looking discount rate for which Tobin's Q is equal to 1 under the classic accumulation rule. This is the correct discount rate to use in co mputing present values in th e household's intertemporal budget constraint. Now, rewrite [ttt028] as

$$\rho_{t+1} = \left(\frac{1+r_t}{1+\pi_{pk,t}}\right) - (1-\delta) = \left(\frac{1+r_t}{g_{pk,t}}\right) - (1-\delta) \qquad \text{[ttt029]}$$

Define the real rate of interest as

$$\tilde{r}_t = \left(\frac{1+r_t}{1+\pi_{pk,t}}\right) - 1 = \left(\frac{1+r_t}{g_{pk,t}}\right) - 1 \qquad \text{[ttt030]}$$

Equation [ttt029] becomes

$$\rho_{t+1} = (1+\tilde{r}_t) - (1-\delta) = \tilde{r}_t + \delta \qquad \text{[ttt031]}$$

This is the user cost of capital: but under the classic accumulation rule, with a one-period lag before investment becomes productive capital, the user cost of capital expected in period $t+1$, ρ_{t+1}, depends on the real forward-looking rate of interest in period t.

In the particular case of static expectations, $\pi_{pk,t} = 0$ and

$$r_t = \rho_t - \delta \qquad \text{[ttt032]}$$

1.4 Solution of the theoretical model

The household's intertemporal optimization problem is

$$\max_{\{c_t\}_{t=\tau}^{T}} U_\tau = \sum_{t=\tau}^{T} \beta^{t-\tau} u(c_t) \qquad \text{[ttt001]}$$

subject to the dynamic budget constraint

$$p_{k,t} k_{t+1} = (1 + \rho_t - \delta) p_{k,t} k_t + y_t - p_{c,t} c_t \,, \; t = \tau, \ldots, T \qquad \text{[ttt018]}$$

and the transversality (no Ponzi game) condition

$$k_{T+1} = \overline{k_{T+1}} \qquad \text{[vvv018]}$$

We shall detail below how, under *truncated rational expectations* (TRE), the growth rates

$$g_{y,\tau} = \frac{y_{\tau+1}}{y_\tau} \qquad \text{[ttt033]}$$

$$g_{pc,\tau} = \frac{p_{c,\tau+1}}{p_{c,\tau}} \qquad \text{[ttt034]}$$

$$g_{pk,\tau} = \frac{p_{k,\tau+1}}{p_{k,\tau}} \qquad \text{[ttt035]}$$

$$g_{\rho,\tau} = \frac{\rho_{\tau+1}}{\rho_\tau} = \frac{w_{k,\tau+1} / w_{k,\tau}}{p_{k,\tau+1} / p_{k,\tau}} = \frac{g_{wk,\tau}}{g_{pk,\tau}} \qquad \text{[ttt036]}$$

are expected to be constant. Under those conditions, it has been demonstrated in Lemelin (2013) that the first-order conditions of that problem yield the following Euler equation for any future period $\theta \geq \tau$:

$$\left(1+g_{\rho,\tau}^{\theta+1-\tau}\rho_\tau-\delta\right)\frac{g_{pk,\tau}}{g_{pc,\tau}}\beta\, u'\left(c_{\theta+1}\right)=u'\left(c_\theta\right) \qquad \text{[ttt037]}$$

The intertemporal budget constraint is obtained by summing the present value of the dynamic (single-period) budget constraint for all periods, beginning with current period τ, and ending with planning horizon T. Applying this to equation [ttt018] yields

$$\sum_{t=\tau}^{T} D_t g_{pc,\tau}^{t-\tau} p_{c,\tau} c_t = p_{k,\tau} k_\tau - D_T g_{pk,\tau}^{T-\tau} p_{k,\tau} \overline{k_{T+1}} + \left(\rho_\tau-\delta\right) p_{k,\tau} k_\tau + \sum_{t=\tau}^{T} D_t g_{y,\tau}^{t-\tau} y_\tau \qquad \text{[ttt038]}$$

where

$$D_t=\left(\frac{1}{g_{pk,\tau}}\right)^{t-\tau}\prod_{\theta=1}^{t-\tau}\left(\frac{1}{\left(1+g_{\rho,\tau}^{\theta}\rho_\tau-\delta\right)}\right) \qquad \text{[ttt039]}$$

with the convention that

$$D_\tau=1 \qquad \text{[ttt040]}$$

D_t is the discount factor, taking into account the expected evolution of the interest rate.

The left-hand side of [ttt038] is the present value of all consumption expenditures to the horizon. This is constrained to be less than the sum of: (1) the surplus of current assets over the present value of terminal assets, (2) current income from capital, and (3) the present value of non-investment income to the horizon. In equilibrium, constraint [ttt038] holds with equality: if the present value of consumption were less than the right-hand side of equation [ttt038], then the household could improve its welfare without violating its budget constraint by raising consumption.

The next step in solving the intertemporal optimization problem is to substitute a specific form of the utility function, here the CRRA utility function

$$u\left(c_t\right)=\frac{c_t^{1-\varsigma}}{1-\varsigma} \qquad \text{[ttt003]}$$

into Euler equation [ttt037], and, by recursion, obtain

$$c_t=D_t^{-\sigma}\left(g_{pc,\tau}^{-1}\beta\right)^{(t-\tau)\sigma}c_\tau \qquad \text{[ttt041]}$$

The solution is obtained by substituting [ttt041] into the intertemporal budget constraint [[ttt038]. After some manipulation, the model solution is given by

$$p_{c,\tau}c_{\tau} = \frac{1}{F_{\tau}}\left[p_{k,\tau}k_{\tau} - D_{T}g_{pk,\tau}^{T-\tau}p_{k,\tau}\overline{k_{T+1}} + (\rho_{\tau} - \delta)p_{k,\tau}k_{\tau} + G_{\tau}y_{\tau}\right] \quad \text{[ttt042]}$$

where

$$F_{\tau} = \sum_{t=\tau}^{T} D_{t}^{1-\sigma} g_{pc,\tau}^{(t-\tau)(1-\sigma)} \beta^{(t-\tau)\sigma} \quad \text{[ttt043]}$$

$$G_{\tau} = \sum_{t=\tau}^{T} D_{t} g_{y,\tau}^{t-\tau} \quad \text{[ttt044]}$$

2. Model

In this section, the theoretical model of section 1 is reformulated to be integrated to an applied CGE model. We start with the PEP-1-t standard recursive dynamic CGE model (Decaluwé *et al.*, 2013), which we modify to make the household savings rate endogenous, based on intertemporal optimization with truncated rational expectations.

In what follows, therefore, we proceed to translate the theoretical model developed in the first section into the PEP-1-t notation. More specifically, we seek to identify the terms of the dynamic budget constraint

$$p_{k,t}k_{t+1} = (1 + \rho_{t} - \delta)p_{k,t}k_{t} + y_{t} - p_{c,t}c_{t} \quad \text{[ttt018]}$$

or, equivalently, of the net savings equation derived from [ttt018]

$$s_{t} = p_{k,t}(k_{t+1} - k_{t}) = (\rho_{t} - \delta)p_{k,t}k_{t} + y_{t} - p_{c,t}c_{t} \quad \text{[ttt019]}$$

2.1 Household income and wealth

The reader is refered to Decaluwé *et al*. (2013) for a detailed description of the PEP-1-t standard model. Here, we shall describe the changes that are brought to the model to implement truncated rational expectations (PEP-1-t-TRE; henceforth PEP-TRE for short).

2.1.1 Household income from capital and wealth

Model equations M 012, M 018, M 023 and M 044 describe the distribution of income from capital. In PEP-TRE, the share parameters are no longer fixed as calibrated, but they evolve through time as agents save and accumulate capital ownership; so the share parameters are now variables, with a time subscript: $\lambda_{h,k,t}^{RK}$.

M 012.	$YHK_{h,t} = \sum_{k} \lambda_{h,k,t}^{RK} \left(\sum_{j} R_{k,j,t} KD_{k,j,t} \right)$
M 018.	$YFK_{f,t} = \sum_{k} \lambda_{f,k,t}^{RK} \left(\sum_{j} R_{k,j,t} KD_{k,j,t} \right)$
M 023.	$YGK_{t} = \sum_{k} \lambda_{gvt,k,t}^{RK} \left(\sum_{j} R_{k,j,t} KD_{k,j,t} \right)$
M 044.	$YROW_{t} = e_{t} \sum_{i} PWM_{i,t} IM_{i,t} + \sum_{l} \lambda_{row,l}^{WL} \left(W_{l,t} \sum_{j} LD_{l,j,t} \right) + \sum_{k} \lambda_{row,k,t}^{RK} \left(\sum_{j} R_{k,j,t} KD_{k,j,t} \right) + \sum_{agd} TR_{row,agd,t}$

where

e_t : Exchange rate[11]: price of foreign currency in terms of local currency

$IM_{i,t}$: Quantity of product *i* imported

$KD_{k,j,t}$: Demand for type *k* capital by industry *j*

$LD_{l,j,t}$: Demand for type *l* labor by industry *j*

$PWM_{i,t}$: World price of imported product *i* (expressed in foreign currency)

$R_{k,j,t}$: Rental rate of type *k* capital in industry *j*

$TR_{ag,agj,t}$: Transfers from agent *agj* to agent *ag*

$W_{l,t}$: Wage rate of type *l* labor

$YFK_{f,t}$: Capital income of type *f* businesses

YGK_t : Government capital income

$YHK_{h,t}$: Capital income of type *h* households

$YROW_t$: Rest-of-the-world income

[11] The default choice of numeraire in PEP-1-t is the exchange rate *e*. This is implemented by fixing the value of *e* as exogenous. But the choice of numeraire in a CGE model is arbitrary (although the interpretation of results can be more or less easy, depending on which numeraire is selected).

As in the theoretical model, the only asset in PEP-TRE is productive capital. It is assumed that capital income shares are equal to capital ownership shares. Consequently, household wealth is defined as

M 116.	$KW_{h,t} = PK_t^{PRI} \sum_{k,j} \left(\lambda_{h,k,t}^{RK} KD_{k,j,t} \right)$

where

$KW_{h,t}$ Stock of capital owned by household h, valued at replacement cost

PK_t^{PRI} : Price of new private capital

The dynamics of ownership shares takes into account depreciation and investment. Investment is financed from pooled savings (equation M 089):

M 089.	$IT_t = \sum_h SH_{h,t} + \sum_f SF_{f,t} + SG_t + SROW_t$

where

IT_t : Total investment expenditures

$SF_{f,t}$: Savings of type f businesses

SG_t : Government savings

$SH_{h,t}$: Savings of type h households

$SROW_t$: Rest-of-the-world savings

So it is reasonable to assume that ownership of the new capital created from investment is distributed in proportion to each agent's savings. The accumulation equation is

M 103.	$KD_{k,j,t+1} = KD_{k,j,t} \left(1 - \delta_{k,j}\right) + IND_{k,j,t}$

where

$\delta_{k,j}$: Depreciation rate of capital k used in industry j

$IND_{k,j,t}$: Volume of new type k capital investment to sector j

The dynamics of capital ownership shares follow

M 121.	$\lambda^{RK}_{h,k,t+1} = \dfrac{\lambda^{RK}_{h,k,t} \sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \left(\dfrac{SH_{h,t}}{IT_t} \right) \sum_j IND_{k,j,t}}{\sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \sum_j IND_{k,j,t}}$
M 122.	$\lambda^{RK}_{f,k,t+1} = \dfrac{\lambda^{RK}_{f,k,t} \sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \left(\dfrac{SF_{f,t}}{IT_t} \right) \sum_j IND_{k,j,t}}{\sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \sum_j IND_{k,j,t}}$
M 123.	$\lambda^{RK}_{gvt,k,t+1} = \dfrac{\lambda^{RK}_{gvt,k,t} \sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \left(\dfrac{SG_{t}}{IT_t} \right) \sum_j IND_{k,j,t}}{\sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \sum_j IND_{k,j,t}}$
M 124.	$\lambda^{RK}_{row,k,t+1} = \dfrac{\lambda^{RK}_{row,k,t} \sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \left(\dfrac{SROW_{t}}{IT_t} \right) \sum_j IND_{k,j,t}}{\sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \sum_j IND_{k,j,t}}$

These equations hold insofar as

iii 001.	$IT_t = PK^{PRI}_t \sum_{k,j} IND_{k,j,t}$

In view of equation M 090,

M 090.	$IT^{PRI}_t = IT_t - IT^{PUB}_t - \sum_i PC_{i,t} \, VSTK_{i,t}$

where

IT^{PRI}_t : Total private investment expenditures

IT^{PUB}_t : Total public investment expenditures

$PC_{i,t}$: Purchaser price of composite comodity *i* (including all taxes and margins)

$VSTK_{i,t}$: Inventory change of commodity *i* this requires that stock variations $VSTK_{i,t}$ be zero, a condition which is met in our application to South Africa.

2.1.2 Household disposable income from capital and other

Household intertemporal optimization concerns consumption and savings, not their transfers to other agents. Now, with the exception of transfers to government, household transfers in PEP-1- t are a fixed proportion of disposable income, as described in model equations M 047 and M 048.

M 047.	$TR_{agng,h,t} = \lambda^{TR}_{agng,h} YDH_{h,t}$
M 048.	$TR_{gvt,h,t} = PIXCON_t^{\eta}\, tr0_{h,t} + tr1_{h,t} YH_{h,t}$

where

η : Price elasticity of indexed transfers and parameters

$PIXCON_t$: Consumer price index

$tr0_{h,t}$: Intercept (transfers by type *h* households to government)

$tr1_{h,t}$: Marginal rate of transfers by type *h* households to government

$YDH_{h,t}$: Disposable income of type *h* households

So, from equation M 015,

M 015.	$CTH_{h,t} = YDH_{h,t} - SH_{h,t} - \sum_{agng} TR_{agng,h,t}$

we define a concept of household disposable income net of transfers as

iii 002.	$YDH_{h,t} - \sum_{agng} TR_{agng,h,t} = CTH_{h,t} + SH_{h,t}$

Household disposable income net of transfers is decomposed into (i) disposable income from capital, defined as

M 111.	$YDHK_{h,t} = \left(1 - ttdh1_{h,t}\right) YHK_{h,t}$

and (ii) disposable income from other sources, net of transfers, defined as

M 112.	$YDHX_{h,t} = \left(1 - ttdh1_{h,t}\right)\left(YH_{h,t} - YHK_{h,t}\right) - ttdh0_{h,t}\, PIXCON_t^{\eta} - \sum_{ag} TR_{ag,h,t}$

where

$ttdh0_{h,t}$: Intercept (income taxes of type *h* households)

$ttdh1_{h,t}$: Marginal income tax rate of type *h* households

Using equation M 048 and

M 047.	$TR_{agng,h,t} = \lambda^{TR}_{agng,h} YDH_{h,t}$

It is easily verified that the sum of the two is indeed $YDH_{h,t} - \sum_{agng} TR_{agng,h,t}$. Combining with

M 015.	$CTH_{h,t} = YDH_{h,t} - SH_{h,t} - \sum_{agng} TR_{agng,h,t}$

where

$CTH_{h,t}$: Consumption budget of type *h* households

we have

iii 003.	$YDHK_{h,t} + YDHX_{h,t} = CTH_{h,t} + SH_{h,t}$

2.2 Rates of return and the interest rate

We define the gross (before taxes and depreciation) household rate of return on capital as

M 113.	$RRK_{h,t} = \frac{YHK_{h,t}}{KW_{h,t}}$

From equations M 012 and M 116, $RRK_{h,t}$ is a weighted average of return rates $\frac{R_{k,j,t}}{PK_t^{PRI}}$,

iii 004.	$RRK_{h,t} = \sum_{k,j} \frac{R_{k,j,t}}{PK_t^{PRI}} \frac{\lambda^{RK}_{h,k,t} KD_{k,j,t}}{\sum_{kj,jj}\left(\lambda^{RK}_{h,kj,jt} KD_{kj,jj,t}\right)}$

where the weights are the shares of capital by type *k* and industry *j* in household *h*'s wealth. We also define the after-tax rate of return as

M 135.	$RHO_{h,t} = (1 - ttdh1_{h,t})RRK_{h,t}$

Finally, applying theoretical equation [ttt028],

$$(1 + r_t) = (1 + \rho_{t+1} - \delta) g_{pk,t} = (1 + \rho_{t+1} - \delta)(1 + \pi_{pk,t}) \quad \text{[ttt028]}$$

we define the interest rate as

M 129.	$IR_t = \left[1 + \frac{\sum_h \left[g_{h,t}^{RRK} RRK_{h,t} KW_{h,t} \right]}{\sum_h KW_{h,t}} - \delta \right] g_t^{PK\_PRI} - 1$

where $g_t^{PK\_PRI}$ and g_t^{RRK} are growth factors. The only difference with [ttt028] is the middle term in the expression between square brackets. In the theoretical model, there is a single household, but in PEP-1-t, there may be more than one (however, in the application to South Africa, there is only one). Consequently, the theoretical rate of return ρ_{t+1} is replaced by a weighted average. The growth factors $g_t^{PK\_PRI}$ and g_t^{RRK} are defined as

M 131. $$g_t^{PK\_PRI} = \frac{PK_{t+1}^{PRI}}{PK_t^{PRI}}$$

M 134. $$g_{h,t}^{RRK} = \frac{RRK_{h,t+1}}{RRK_{h,t}}$$

These are to be explained below.

The definition of interest rate IR_t is a significant difference in PEP-TRE relative to the basic version of PEP-1-t. In the latter, the rate of interest is merely a rationing device that equates the demand for investment with the amount of savings. It has no other role in the model. Here, however, it must be consistent with the rate of return on household wealth: it is indeed the rate of interest that households must receive to be persuaded to invest their savings. Since the rate of interest is no longer free to play its role as a rationing device, that role must be assumed by another variable. Consequently, the scale parameter in the investment equation becomes a scale variable[12]:

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[12] The reader familiar with PEP-1-t will have noticed that in the latter, ϕ is indexed in k and j. In practice however, the PEP-1-t calibration procedure results in uniform values for $\phi_{k,j}$. Under those circumstances, it can be shown that it is indifferent to use the interest rate or ϕ as the savings-investment equilibrating (rationing) device. See Appendix for details.

M 108.	$IND_{k,bus,t} = \phi_t \left[\frac{R_{k,bus,t}}{U_{k,bus,t}} \right]^{\sigma_{k,bus}^{INV}} KD_{k,bus,t}$

where

ϕ_t : Scale parameter (allocation of investment to industries)

$U_{k,j,t}$: User cost of type *k* capital in industry *j*

2.3 Household savings

We now proceed to identify the theoretical concepts in the net savings equation derived from [ttt018]

$$s_t = p_{k,t}(k_{t+1} - k_t) = (\rho_t - \delta)p_{k,t}k_t + y_t - p_{c,t}c_t \quad \text{[ttt019]}$$

Reorganize [iii 003] as

iii 005.	$SH_{h,t} = YDHK_{h,t} + YDHX_{h,t} - CTH_{h,t}$

parallel to [ttt019]. The savings concept represented by the left-hand side of [iii 005], however, is *gross* household savings (*ghs*). The corresponding theoretical variable would be

$$ghs_t = s_t + \delta\ p_{k,t}k_t = p_{k,t}(k_{t+1} - k_t) + \delta\ p_{k,t}k_t = \rho_t p_{k,t}k_t + y_t - p_{c,t}c_t \quad \text{[ttt045]}$$

$$ghs_t = s_t + \delta\ p_{k,t}k_t = p_{k,t}[k_{t+1} - (1-\delta)k_t] = \boxed{\rho_t p_{k,t}k_t + y_t - p_{c,t}c_t} \quad \text{[ttt046]}$$

Its equivalent in PEP-1-t is found by combining equations M 111, M 113 and M 135,

M 111.	$YDHK_{h,t} = (1 - ttdh1_{h,t})YHK_{h,t}$
M 113.	$RRK_{h,t} = RRK_{h,t}^{DIR} = \frac{YHK_{h,t}}{KW_{h,t}^{DIR}}$
M 135.	$RHO_{h,t} = (1 - ttdh1_{h,t})RRK_{h,t}$

and substituting into [iii 005] to obtain

iii 006.	$SH_{h,t} = RHO_{h,t}KW_{h,t} + YDHX_{h,t} - CTH_{h,t}$

2.4 Household wealth accumulation

Let us now concentrate on the definition of gross savings in terms of wealth accumulation, as expressed in the first two equalities of [ttt046]

$$ghs_t = s_t + \delta\ p_{k,t} k_t = p_{k,t}\left[k_{t+1} - (1-\delta)k_t\right] = \rho_t p_{k,t} k_t + y_t - p_{c,t} c_t \qquad \text{[ttt046]}$$

In the equation above, gross household savings are defined as the difference between the stock of capital owned in t+1, k_{t+1}, and the stock of capital owned in t after depreciation, $(1-\delta)\,k_t$, both valued at the current price in period t. To relate these to PEP-1-t variables, we begin with the definition of household wealth given above,

M 116.	$KW_{h,t} = PK_t^{PRI} \sum_{k,j} \left(\lambda_{h,k,t}^{RK} KD_{k,j,t} \right)$

Using

M 121.	$\lambda_{h,k,t+1}^{RK} = \dfrac{\lambda_{h,k,t}^{RK} \sum_{j} \left[\left(1-\delta_{k,j}\right) KD_{k,j,t} \right] + \left(\dfrac{SH_{h,t}}{IT_t} \right) \sum_{j} IND_{k,j,t}}{\sum_{j} \left[\left(1-\delta_{k,j}\right) KD_{k,j,t} \right] + \sum_{j} IND_{k,j,t}}$

we can write, from equation M 116,

iii 007.	$KW_{h,t+1} = PK_{t+1}^{PRI} \sum_{k,j} \left(\dfrac{\lambda_{h,k,t}^{RK} \sum_{jj} \left[\left(1-\delta_{k,jj}\right) KD_{k,jj,t} \right] + \left(\dfrac{SH_{h,t}}{IT_t} \right) \sum_{jj} IND_{k,jj,t}}{\sum_{jj} \left[\left(1-\delta_{k,jj}\right) KD_{k,jj,t} \right] + \sum_{jj} IND_{k,jj,t}} KD_{k,j,t+1} \right)$
iii 008.	$KW_{h,t+1} = PK_{t+1}^{PRI} \sum_{k} \left(\dfrac{\lambda_{h,k,t}^{RK} \sum_{jj} \left[\left(1-\delta_{k,jj}\right) KD_{k,jj,t} \right] + \left(\dfrac{SH_{h,t}}{IT_t} \right) \sum_{jj} IND_{k,jj,t}}{\sum_{jj} \left[\left(1-\delta_{k,jj}\right) KD_{k,jj,t} \right] + \sum_{jj} IND_{k,jj,t}} \sum_{j} KD_{k,j,t+1} \right)$

Substituting from equation M 103,

M 103.	$KD_{k,j,t+1} = KD_{k,j,t} \left(1-\delta_{k,j}\right) + IND_{k,j,t}$

we find

iii 009.	$KW_{h,t+1} = PK_{t+1}^{PRI} \sum_k \left[\lambda_{h,k,t}^{RK} \sum_{jj} \left[\left(1 - \delta_{k,jj} \right) KD_{k,jj,t} \right] + \left(\frac{SH_{h,t}}{IT_t} \right) \sum_{jj} IND_{k,jj,t} \right]$

We now introduce the simplifying assumption that all rates of depreciation are equal, so that

iii 010.	$KW_{h,t+1} = (1-\delta) PK_{t+1}^{PRI} \sum_{k,j} \lambda_{h,k,t}^{RK} KD_{k,j,t} + PK_{t+1}^{PRI} \left(\frac{SH_{h,t}}{IT_t} \right) \sum_{k,j} IND_{k,j,t}$

Using wealth definition M 116, together with equation [iii 001]

iii 001.	$IT_t = PK_t^{PRI} \sum_{k,j} IND_{k,j,t}$

and substituting into [iii010], we obtain

iii 011.	$KW_{h,t+1} = (1-\delta) \frac{PK_{t+1}^{PRI}}{PK_t^{PRI}} KW_{h,t} + SH_{h,t}$
iii 012.	$SH_{h,t} = \frac{PK_t^{PRI}}{PK_{t+1}^{PRI}} KW_{h,t+1} - (1-\delta) KW_{h,t}$

Indeed, in accordance with theoretical equation [ttt046], $SH_{h,t}$ is gross household savings.

2.5 Household dynamic budget constraint

From [iii 012], substitute for $SH_{h,t}$ into

iii 006.	$SH_{h,t} = RHO_{h,t} KW_{h,t} + YDHX_{h,t} - CTH_{h,t}$

using

M 131.	$g_t^{PK\_PRI} = \frac{PK_{t+1}^{PRI}}{PK_t^{PRI}}$

and there results

iii 013.	$\frac{1}{g_t^{PK\_PRI}} KW_{h,t+1} - (1-\delta) KW_{h,t} = RHO_{h,t} KW_{h,t} + YDHX_{h,t} - CTH_{h,t}$
iii 014.	$CTH_{h,t} = \left(1 + RHO_{h,t} - \delta\right) KW_{h,t} + YDHX_{h,t} - \frac{1}{g_t^{PK\_PRI}} KW_{h,t+1}$

Equation [iii 014] is the household dynamic budget constraint. It is the PEP-1-t equivalent of

$$p_{c,t} c_t = \left(1 + \rho_t - \delta\right) p_{k,t} k_t + y_t - p_{k,t} k_{t+1} \quad \text{[ttt047]}$$

which is the theoretical dynamic budget constraint derived from

$$s_t = p_{k,t} \left(k_{t+1} - k_t\right) = \left(\rho_t - \delta\right) p_{k,t} k_t + y_t - p_{c,t} c_t \quad \text{[ttt019]}$$

2.6 Truncated rational expectations

Until now, we have not specified expectations. We now proceed with the model under truncated rational expectations (TRE). In the TRE framework, households have rational expectations for the current period and the following one. Accordingly, the model is solved simultaneously for two periods at a time, the current period τ and the following period $\tau+1$. Household (rational) expectations for period $\tau+1$ are given by the model solution. For subsequent periods, household expectations are formed by extrapolating from τ and $\tau+1$ solution values, assuming a constant rate of change. With these extrapolations, the intertemporal problem to any planning horizon T is entirely endogenous to the two-period model.

So the model is solved iteratively for successive pairs of periods: solve for periods 0 and 1 and keep the period 0 solution; then solve for periods 1 and 2 and keep period 1 solution; next solve for periods 2 and 3, etc. For each pair [$\tau, \tau+1$] of periods, the household solves its intertemporal optimization problem up to its planning horizon (the household planning horizon moves forward one period each period). But only the first period (τ) of the program actually gets implemented. For $\tau+1$, the household applies the first period of the optimal intertemporal program computed as part of the simultaneous [$\tau+1, \tau+2$] model solution. Etc.

The extrapolation formulae that generate household expectations are:

iii 015.	$YDHX_{h,t} = g_{h,\tau}^{YDHX} YDHX_{h,t-1} = \left(g_{h,\tau}^{YDHX}\right)^{t-\tau} YDHX_{h,\tau}$, for $t \geq \tau$
iii 016.	$PIXCON_t = g_\tau^{PIXCON} PIXCON_{t-1} = \left(g_\tau^{PIXCON}\right)^{t-\tau} PIXCON_\tau$, for $t \geq \tau$

iii 017.	$PK_{t}^{PRI} = g_{\tau}^{PK\_PRI} PK_{t-1}^{PRI} = \left(g_{\tau}^{PK\_PRI}\right)^{t-\tau} PK_{\tau}^{PRI}$, for $t \geq \tau$
iii 018.	$RHO_{h,t} = g_{h,\tau}^{RHO} RHO_{h,t-1} = \left(g_{h,\tau}^{RHO}\right)^{t-\tau} RHO_{h,\tau}$, for $t \geq \tau$

where the growth rates are simply

M 130.	$g_{h,\tau}^{YDHX} = \dfrac{YDHX_{h,\tau+1}}{YDHX_{h,\tau}}$
M 131.	$g_{\tau}^{PK\_PRI} = \dfrac{PK_{\tau+1}^{PRI}}{PK_{\tau}^{PRI}}$
M 132.	$g_{\tau}^{PC} = \dfrac{PIXCON_{\tau+1}}{PIXCON_{\tau}}$
M 133.	$g_{h,\tau}^{RHO} = \dfrac{RHO_{h,\tau+1}}{RHO_{h,\tau}}$

It is important to note that, under this formulation, the household's planning horizon is entirely independent from the length of the model simulation run. For that reason, the parameter *LifeEnd* is the number of periods beyond the current one over which households optimize their consumption. The index *Lifetime* (alias *Lifetoo*) designates successive periods of the intertemporal optimization time-span:

$Lifetime = 1,...,LifeEnd$

2.7 Model solution

Following the procedure outlined in 1.4 above, we obtain the Euler equation

aaa 080.	$CTH_{h,t+Lifetime+1}^{Plan} = \left\{ \beta \dfrac{g_{t}^{PK\_PRI}}{g_{t}^{PIXCON}} \left[1 + \left(g_{h,t}^{RHO}\right)^{Lifetime+1} RHO_{h,t} - \delta \right] \right\}^{\sigma} g_{t}^{PIXCON} CTH_{h,t+Lifetime}^{Plan}$

where

$Lifetime = 1,...,LifeEnd$

and $CTH^{Plan}_{h,t+Lifetime}$ represents the amount consumption expenditures *planned* at time t for future period $t+Lifetime$. And, by recursion, we obtain the following equation,

iii 019.

$$CTH^{Plan}_{h,t+Lifetime} = \left[\prod_{\theta=1}^{Lifetime}\left(1+\left(g^{RHO}_{h,t}\right)^{\theta} RHO_{h,t}-\delta\right)\right]^{\sigma}\left(\beta\frac{g^{PK\_PRI}_{t}}{g^{PIXCON}_{t}}\right)^{\sigma\ Lifetime}\left(g^{PIXCON}_{t}\right)^{Lifetime} CTH_{h,t}$$

in which the optimal amount of consumption expenditures for any future period up to the household planning horizon is expressed in terms of current values of variables and growth rates. Let

M 125.

$$D_{h,Lifetime,t} = \left(\frac{1}{g^{PK\_PRI}_{t}}\right)^{Lifetime}\left[\prod_{\theta=1}^{Lifetime}\left(\frac{1}{1+\left(g^{RHO}_{h,t}\right)^{\theta} RHO_{h,t}-\delta}\right)\right]$$

and [iii 019] becomes

iii 020.

$$CTH^{Plan}_{h,t+Lifetime} = \left(D_{h,Lifetime,t}\right)^{-\sigma}\left(\frac{\beta}{g^{PIXCON}_{t}}\right)^{\sigma\ Lifetime}\left(g^{PIXCON}_{t}\right)^{Lifetime} CTH_{h,t}$$

In the two-period sim ultaneous solution of the m odel, equation [iii 020] applied to the second period (*Lifetime*=1) is

M 136.

$$CTH^{Plan}_{h,t+1} = \left(D_{h,1,t}\right)^{-\sigma}\left(\frac{\beta}{g^{PIXCON}_{t}}\right)^{\sigma} g^{PIXCON}_{t} CTH_{h,t}$$

The intertemporal budget constraint at time t is given by

iii 021.

$$CTH_{h,t} + \sum_{Lifetime} D_{h,Lifetime,t} CTH^{Plan}_{h,t+Lifetime} = \left(1+RHO_{h,t}-\delta\right)KW_{h,t} + \left[1+\sum_{Lifetime} D_{h,Lifetime,t}\left(g^{YDHX}_{h,t}\right)^{Lifetime}\right]YDHX_{h,t} - \frac{D_{h,LifeEnd,t}}{g^{PK\_PRI}_{t}} KW^{TERM}_{h,t}$$

where

$KW_{h,t}^{TERM}$: Terminal wealth used in intertemporal optimization at time t by household h; it is the amount of wealth which, according to its plans at time t, the households expects to leave at the end of period $t + LifeEnd$.

This variable defines the transversality (No Ponzi game) condition for each period's intertermporal optimization exercise. Substitute [iii 020] into intertemporal budget constraint [iii 021] and find

iii 022.	$CTH_{h,t} + \sum_{Lifetime} \left(D_{h,Lifetime,t}\right)^{1-\sigma} \left(\frac{\beta}{g_t^{PIXCON}}\right)^{\sigma\, Lifetime} \left(g_t^{PIXCON}\right)^{Lifetime} CTH_{h,t} = \left(1 + RHO_{h,t} - \delta\right) KW_{h,t} + \left[1 + \sum_{Lifetime} D_{h,Lifetime,t} \left(g_{h,t}^{YDHX}\right)^{Lifetime}\right] YDHX_{h,t} - \frac{D_{h,LifeEnd,t}}{g_t^{PK\_PRI}} KW_{h,t}^{TERM}$

Let

M 126.	$Z1_{h,t} = 1 + \sum_{lifetime=1}^{LifeEnd} D_{h,lifetime,t} \left(g_{h,t}^{YDHX}\right)^{lifetime}$
M 127.	$Z2_{h,t} = 1 + \sum_{lifetime=1}^{EndLife} \left(D_{h,lifetime,t}\right)^{1-\sigma} \left(\frac{\beta}{g_t^{PIXCON}}\right)^{\sigma\, lifetime} \left(g_t^{PIXCON}\right)^{lifetime}$

and [iii 022] becomes

M 128.	$CTH_{h,t} = \frac{1}{Z2_{h,t}} \left[\left(1 + RHO_{h,t} - \delta\right) KW_{h,t} + Z1_{h,t} YDHX_{h,t} - \frac{D_{h,LifeEnd,t}}{g_t^{PK\_PRI}} KW_{h,t}^{TERM}\right]$

This is the new consumption equation in PEP-TRE. Equation M 128, however, applies only to the first of the moving two-period simultaneous solution of the model. For second-period consumption expenditures, equation M 136 is used. Applying M 128 to the second period yields a solution that is only slightly different, but is inconsistent with the view that households make their decisions in the first period with perfect foresight regarding the second.

What remains to be defined is how terminal wealth (the transversality condition) is determined. In the current version of PEP-TRE, we assume that the household wants terminal wealth *per capita* to be equal to its initial wealth in real terms. Here, « in real terms » is to be understood as meaning with the same consumer purchasing power.

M 120.	$KW_{h,t}^{TERM} = pop_t KW_{h,t}^{O} PIXCON_t \left(g_t^{PIXCON}\right)^{Lifetime}$

2.8 Model summary

Table 1 below summarizes the differences in PEP-TRE relative to the standard version of PEP1-t. Red markings highlight changes that are visually less obvious.

Table 1 – differences in PEP-TRE relative to the standard version of PEP1-t

	PEP-1-t-TRE		PEP-1-t
M 012.	$YHK_{h,t} = \sum_k \lambda^{RK}_{h,k,t} \left(\sum_j R_{k,j,t} KD_{k,j,t} \right)$	M 012.	$YHK_{h,t} = \sum_k \lambda^{RK}_{h,k} \left(\sum_j R_{k,j,t} KD_{k,j,t} \right)$
M 016.	$SH_{h,t} = YDH_{h,t} - CTH_{h,t} - \sum_{agng} TR_{agng,h,t}$	M 016.	$SH_{h,t} = PIXCON_t^{\eta} sh0_{h,t} + sh1_{h,t} YDH_{h,t}$
M 018.	$YFK_{f,t} = \sum_k \lambda^{RK}_{f,k,t} \left(\sum_j R_{k,j,t} KD_{k,j,t} \right)$	M 018.	$YFK_{f,t} = \sum_k \lambda^{RK}_{f,k} \left(\sum_j R_{k,j,t} KD_{k,j,t} \right)$
M 023.	$YGK_t = \sum_k \lambda^{RK}_{gvt,k,t} \left(\sum_j R_{k,j,t} KD_{k,j,t} \right)$	M 023.	$YGK_t = \sum_k \lambda^{RK}_{gvt,k} \left(\sum_j R_{k,j,t} KD_{k,j,t} \right)$
M 044.	$YROW_t = e_t \sum_i PWM_{i,t} IM_{i,t} + \sum_l \lambda^{WL}_{row,l} \left(W_{l,t} \sum_j LD_{l,j,t} \right) + \sum_k \lambda^{RK}_{row,k,t} \left(\sum_j R_{k,j,t} KD_{k,j,t} \right) + \sum_{agd} TR_{row,agd,t}$	M 044.	$YROW_t = e_t \sum_i PWM_{i,t} IM_{i,t} + \sum_k \lambda^{RK}_{row,k} \left(\sum_j R_{k,j,t} KD_{k,j,t} \right) + \sum_{agd} TR_{row,agd,t}$
M 108.	$IND_{k,bus,t} = \phi_t \left[\frac{R_{k,bus,t}}{U_{k,bus,t}} \right]^{\sigma^{INV}_{k,bus}} KD_{k,bus,t}$	M 108.	$IND_{k,bus,t} = \phi_{k,bus} \left[\frac{R_{k,bus,t}}{U_{k,bus,t}} \right]^{\sigma^{INV}_{k,bus}} KD_{k,bus,t}$
M 111.	$YDHK_{h,t} = \left(1 - ttdh1_{h,t}\right) YHK_{h,t}$		**ABSENT**
M 112.	$YDHX_{h,t} = \left(1 - ttdh1_{h,t}\right)\left(YH_{h,t} - YHK_{h,t}\right) - ttdh0_{h,t} PIXCON_t^{\eta} - \sum_{ag} TR_{ag,h,t}$		**ABSENT**
M 113.	$RRK_{h,t} = \frac{YHK_{h,t}}{KW_{h,t}}$		**ABSENT**

Table 1 (continued)

PEP-1-t-TRE		PEP-1-t	
M 116.	$KW_{h,t} = PK_t^{PRI} \sum_{k,j} \left(\lambda_{h,k,t}^{RK} KD_{k,j,t} \right)$		**ABSENT**
M 120.	$KW_{h,t}^{TERM} = pop_t KW_{h,t}^{O} PIXCON_t \left(g_t^{PIXCON} \right)^{Lifetime}$		**ABSENT**
M 121.	$\lambda_{h,k,t+1}^{RK} = \dfrac{\lambda_{h,k,t}^{RK} \sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \left(\dfrac{SH_{h,t}}{IT_t} \right) \sum_j IND_{k,j,t}}{\sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \sum_j IND_{k,j,t}}$		**ABSENT**
M 122.	$\lambda_{f,k,t+1}^{RK} = \dfrac{\lambda_{f,k,t}^{RK} \sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \left(\dfrac{SF_{f,t}}{IT_t} \right) \sum_j IND_{k,j,t}}{\sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \sum_j IND_{k,j,t}}$		**ABSENT**
M 123.	$\lambda_{gvt,k,t+1}^{RK} = \dfrac{\lambda_{gvt,k,t}^{RK} \sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \left(\dfrac{SG_t}{IT_t} \right) \sum_j IND_{k,j,t}}{\sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \sum_j IND_{k,j,t}}$		**ABSENT**
M 124.	$\lambda_{row,k,t+1}^{RK} = \dfrac{\lambda_{row,k,t}^{RK} \sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \left(\dfrac{SROW_t}{IT_t} \right) \sum_j IND_{k,j,t}}{\sum_j \left[\left(1-\delta_{k,j} \right) KD_{k,j,t} \right] + \sum_j IND_{k,j,t}}$		**ABSENT**

Table 1 (continued)

PEP-1-t-TRE		PEP-1-t	
M 125.	$D_{h,Lifetime,t} = \left(\frac{1}{g_t^{PK\_PRI}}\right)^{Lifetime} \left[\prod_{\theta=1}^{Lifetime}\left(\frac{1}{1+\left(g_{h,t}^{RHO}\right)^{\theta} RHO_{h,t} - \delta}\right)\right]$		**ABSENT**
M 126.	$Z1_{h,t} = 1 + \sum_{lifetime=1}^{LifeEnd} D_{h,lifetime,t} \left(g_{h,t}^{YDHX}\right)^{lifetime}$		**ABSENT**
M 127.	$Z2_{h,t} = 1 + \sum_{lifetime=1}^{EndLife} \left(D_{h,lifetime,t}\right)^{1-\sigma} \left(\frac{\beta}{g_t^{PIXCON}}\right)^{\sigma\ lifetime} \left(g_t^{PIXCON}\right)^{lifetime}$		**ABSENT**
M 128.	$CTH_{h,t} = \frac{1}{Z2_{h,t}}\left[\left(1+RHO_{h,t} - \delta\right)KW_{h,t} + Z1_{h,t} YDHX_{h,t} - \frac{D_{h,LifeEnd,t}}{g_t^{PK\_PRI}} KW_{h,t}^{TERM}\right]$	M 015.	$CTH_{h,t} = YDH_{h,t} - SH_{h,t} - \sum_{agng} TR_{agng,h,t}$
M 136.	$CTH_{h,t+1}^{Plan} = \left(D_{h,1,t}\right)^{-\sigma}\left(\frac{\beta}{g_t^{PIXCON}}\right)^{\sigma} g_t^{PIXCON} CTH_{h,t}$		**ABSENT**
M 129.	$IR_t = \left[1 + \frac{\sum_h \left[g_{h,t}^{RRK} RRK_{h,t} KW_{h,t}\right]}{\sum_h KW_{h,t}} - \delta\right] g_t^{PK} - 1$		**ABSENT**
M 130.	$g_{h,\tau}^{YDHX} = \frac{YDHX_{h,\tau+1}}{YDHX_{h,\tau}}$		**ABSENT**
M 131.	$g_t^{PK\_PRI} = \frac{PK_{t+1}^{PRI}}{PK_t^{PRI}}$		**ABSENT**

Table 1 (continued)

M 132.	$g_t^{PIXCON} = \frac{PIXCON_{t+1}}{PIXCON_t}$		**ABSENT**
M 133.	$g_{h,t}^{RHO} = \frac{RHO_{h,t+1}}{RHO_{h,t}}$		**ABSENT**
M 134.	$g_{h,t}^{RRK} = \frac{RRK_{h,t+1}}{RRK_{h,t}}$		**ABSENT**
M 135.	$RHO_{h,t} = (1 - ttdh1_{h,t})RRK_{h,t}$		**ABSENT**

3. Application to South Africa, 2005

3.1 SAM

The PEP-TRE model is applied to South Africa. Specifically, we use the 2005 South African SAM by Davies and Thurlow (2011) [13]. The SAM was first converted to the PEP-1-t format and aggregated (see appendix for the list of industries/commodities).

Next, inventory changes in the SAM were eliminated. This was not done only for convenience. Indeed, inventory variations are notoriously volatile, and for that reason difficult to model in a CGE. In particular, if inventory changes are to be modelled as a particular form of investment, then, no matter how reasonable the model may be, its calibration is highly dependent on business cycle conditions at the moment the SAM was constructed. Modellers often choose to fix inventory variations exogenously, but the idea that surplus production can be dumped indefinitely to inventory stretches the imagination, and even more so the idea that supply can be pumped indefinitely out of inventories. To eliminate inventory variations, using PEP-1-1, we conducted a static simulation of the model in which inventory variation is exogenously fixed at zero. The resulting solution was then used to construct a new base-year SAM[14].

We use two variants of the SAM. The first is the original Davies and Thurlow SAM, except for the elimination of inventory variations. But in that SAM, the household savings rate is very low (2.6% of disposable income net of transfers to other agents[15]). The objective in constructing a second variant of the SAM was to have an example, albeit artificial, of an economy where the household savings rate is high, in order to determine the implications of the initial savings rate on the endogenous evolution of the savings rate in the model. We constructed an alternate SAM, with a household savings rate of 16.8%. This was achieved by setting firm savings in the original SAM to zero, and increasing household savings by the same amount, while balancing accounts by adding an equivalent transfer from firms to households (these transfers may be interpreted as dividends). The resulting SAM is our second variant, which is identical to the original one in every other aspect. The two variants are nicknamed LoSH (low household savings) and HiSH (high household savings).

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13 Thanks to Hélène Maisonnave, who has kindly transmitted to us the 2005 South African SAM. That SAM is no longer available on the IFPRI website; it has been replaced by a 2009 SAM. The reason for using the 2005 one will be explicited shortly.

14 This is in fact the reason why PEP-TRE was not applied to Senegal as initially planned. When inventory changes were eliminated using the same technique for Senegal, the resulting SAM was so radically different from the initial one that it could hardly be considered to represent the same economy. As a matter of fact, inventory changes in the original Senegal SAM were so important that the model had to be solved several times to eliminate them in slices.

3.2 Parametrization

Parametrizing PEP-TRE poses quite a challenge. From the model summary in Table 1, it can be seen that the five growth factors defined in equations M 130-M 134 appear critically in the household intertemporal optimization equations M 120 and M 125-M 129. With a single observation year (the SAM), these growth rates are unknown, and with unknown growth factors, the model cannot be calibrated. Moreover, the intertemporal rate of substitution σ and the psychological discount factor $\beta = 1/(1+\psi)$, which appear in equation M 127, are free parameters (as are, for instance, CES elasticities of substitution).

We applied a parametrization procedure that can be summarized as follows.

1. Set the values of all growth factors provisionally to 1.
2. After everything else has been calibrated, compute $Z2_h^O$ by inverting equation M 128:

iii 023.	$Z2_{h,t} = \frac{1}{CTH_{h,t}}\left[\left(1+RHO_{h,t}-\delta\right)KW_{h,t} + Z1_{h,t}YDHX_{h,t} - \frac{D_{h,LifeEnd,t}}{g_t^{PK\_PRI}}KW_{h,t}^{TERM}\right]$

3. Set the intertemporal rate of substitution from the literature; here we use $\sigma = 0.35$.
4. Solve equation M 127 as an implicit equation for β.

At that point, the model is fully parametrized for growth factors equal to 1. Then,

5. Solve the model simultaneously for periods 1 and 2 (2005 and 2006).
6. The model solution will be consistent with the first-year values, but the growth factors computed from the second-year solution will be different from their provisional values.
7. Fix the growth factors at their solution values and re-calibrate all variables that depend on them.
8. Return to step 5 and repeat until the solution values of the growth factors are equal to their provisional values.

The procedure may require slicing the adjustments in step 7. When slicing, rather than setting the growth factors at their solution values, they are fixed as a linear combination of their previous values and their solution values; in some instances, the weight of the solution value had to be as low as 0.1. In the application to South Africa, this procedure, with slicing, generally necessitated between 35 and 40 solutions before converging.

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15 $SH_h^O / \left(YDHK_h^O + YDHX_h^O\right)$

3.3 Closure

The model closure is standard. The exchange rate is the numeraire. The current account balance is fixed exogenously and grows at the same rate as population. The same applies to the labor supply, government savings, and public investment. The rate of growth of South African population is set at 1.34%[16].

3.4 Simulations

The BAU scenario runs over a 50-year span, to horizon 2054. The household planning horizon is set at 30 years. Given the importance of mining in South Africa, the first "counter-factual" simulation consists in a permanent 50% drop in the (exogenous) international price of minerals (products of the industry labeled *Mining* in our aggregation).

Our second simulation is to test household reaction to a sudden unexpected and substantial reduction in wealth. It is expected that, faced with such circumstances, with unchanged transversality condition, households will try to restore their level of wealth and, to do so, will increase their savings. This indeed is how many have interpreted the rise in the U.S. household savings rate after the bottom fell out of the real estate market in 2008. To simulate a sudden decline in wealth, the Alice-in-Wonderland shock that was inflicted to households in the model is a confiscation of 20% of their wealth by government. As we shall see, results were not those expected.

In the third simulation, we test the impact on household savings of a reduction in the rate of return on wealth. It is expected that a lower rate of return is a disincentive for savings. This experiment consists in slapping a 2% surtax on houseshold capital income in the LoSH case, and a 10% surtax in the HiSH case. And this time, the result is as expected.

3.5 Results

The first result to point out is... that the model actually runs! And it appears to be quite robust. This required several programming adjustments, but the apparent ease of solution exceeds this author's expectations.

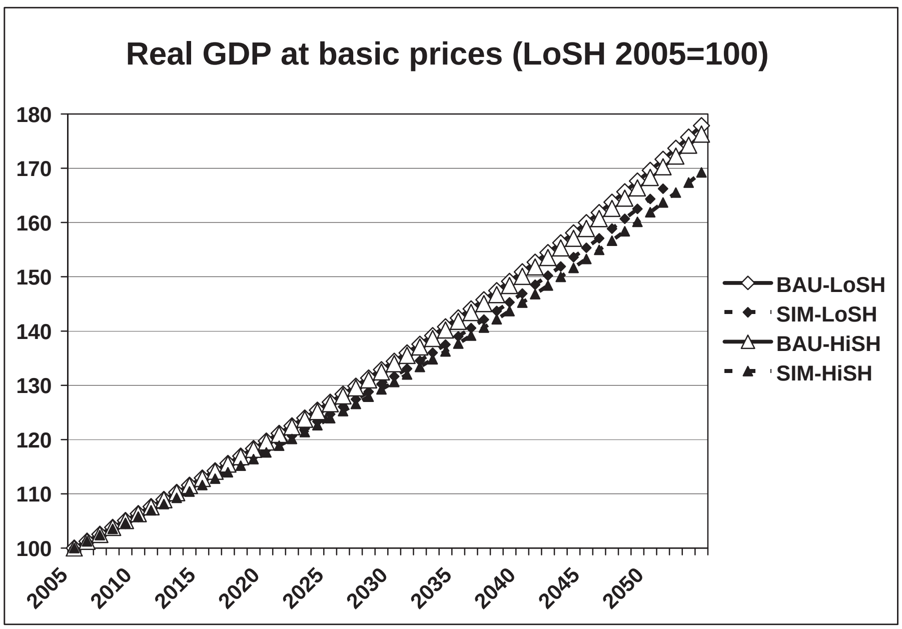
3.5.1 Simulation 1: permanent 50% drop in the international price of minerals

Figure 1 displays the evolution of real GDP at basic prices for each of the four scenarios:

16 http://en.wikipedia.org/wiki/Demographics_of_South_Africa

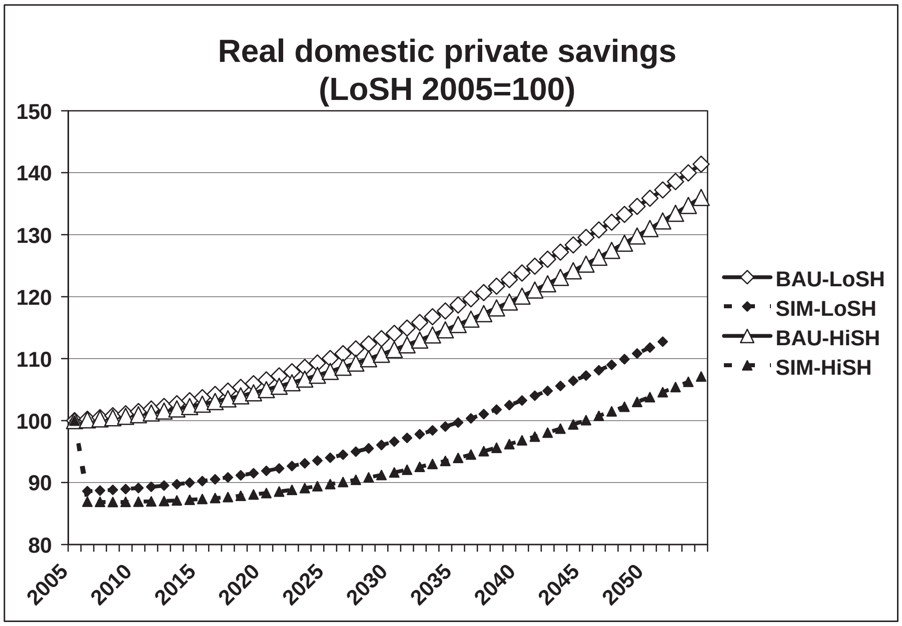
- BAU-LoSH is the reference scenario with the original Davies and Thurlow SAM, where the household savings rate is low (LoSH);
- SIM-LoSH is the scenario in which the world price of minerals falls by 50% in 2006, with the original Davies and Thurlow SAM, where the household savings rate is low (LoSH);
- BAU-HiSH is the reference scenario with the alternate SAM, where the household savings rate is high (HiSH);
- SIM-HiSH is the scenario in which the world price of minerals falls by 50% in 2006, with the alternate SAM, where the household savings rate is high (HiSH).

Figure 1



Since labor supply is identical in all four scenarios, and given that real GDP is essentially a measure of the volume of primary factors, then the differences must come from the volume of capital, and more specifically from capital accumulation through (savings-driven) investment. With fixed government savings and current account balance (foreign savings), differences in the volume of capital can only result from differences in real domestic private savings (household and firm savings, divided by the price of capital). This is shown in Figure 2.

Figure 2



At first sight, one might be surprised that domestic private savings are lower in the high household savings rate variant (HiSH). Recall however that, in the HiSH variant, firm savings are zero: the amount that firms would have saved is transferred to households, whose income is consequently higher, leading to more savings. But whereas the firms' contribution to domestic savings in the LoSH variant (SF) is mechanically determined as a fraction of firms' disposable income, household savings in both variants are subject to their intertemporal optimization. So what Figure 2 shows is that, when households receive from firms a transfer that is equivalent to what the latter would have saved, they choose to spend part of it on consumption. And that occurs in spite of the fact that the HiSH variant of the model is calibrated so that household savings are initially equal to total domestic private savings in the LoSH variant.

Figure 3 shows the evolution of real household disposable income, net of transfers. In constructing the second variant of the SAM, household income was artificially boosted by a transfer from firms equal to their savings in the original SAM. Consequently, household income is higher in the second variant of the SAM and in the HiSH scenarios. In both pairs of scenarios (Lo- and HiSH), the shock on the world price of minerals has a negative impact on household income. But thereafter, household incomes resume their ascent.

Figure 3

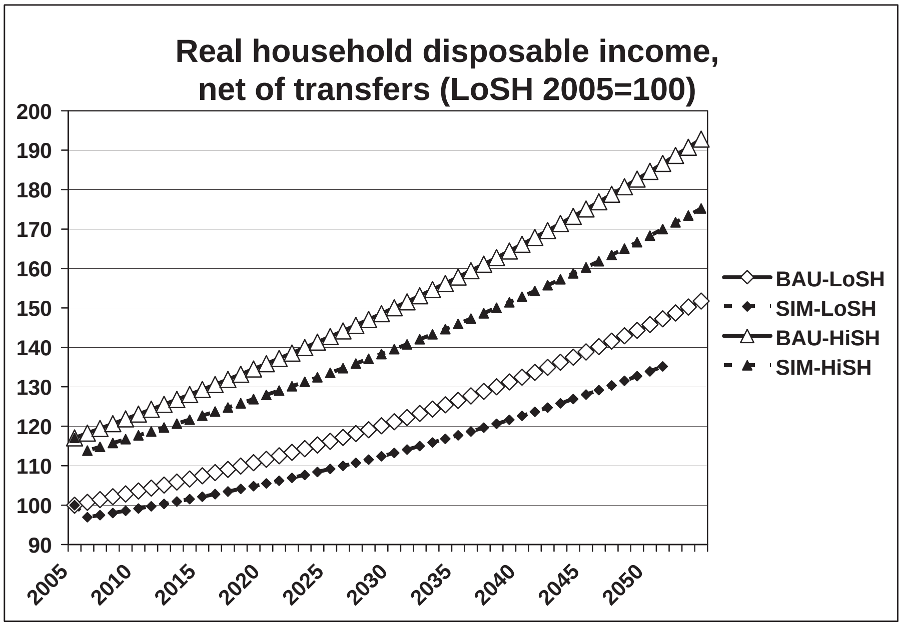
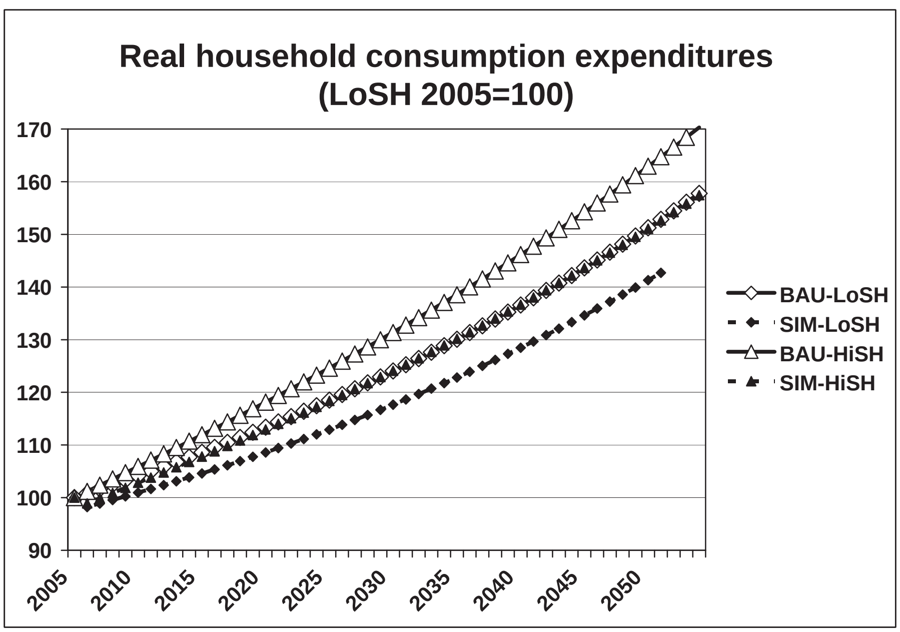


Figure 4 displays the evolution of real household consumption expenditures.

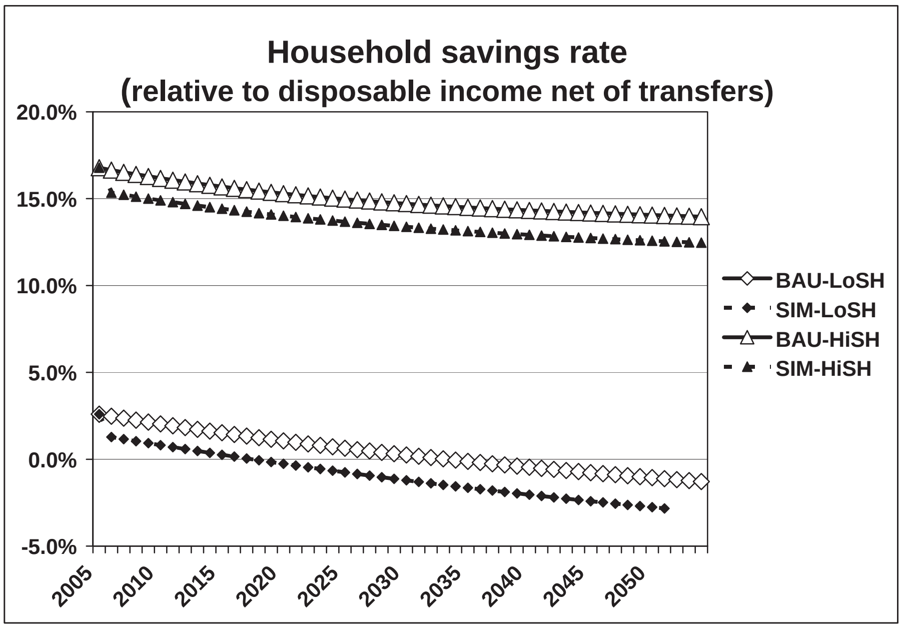
Figure 4



Bringing together Figures 2-4 shows that the income shock of a drop in the international price of minerals is absorbed mostly by a drop in savings, as households reschedule their lifetime savings-consumption plan[17]. It can be seen that they do not revert progressively to their original plan, but rather settle for a lower consumption and savings regime after the shock. This is consistent with the model construct, according to which households solve their intertemporal optimization problem afresh every period, on the basis of their currently held expectations, which are projections from a two-period near-perfect foresight.

In Figure 5, it is seen that indeed, both in the high- and in the low-savings rate situation, the savings rate remains indefinitely below what it would have been without the shock on the international price of minerals. Moreover, in all scenarios, the savings rate falls over time, and in the two LoSH scenarios, it becomes negative.

Figure 5



Finally, let us consider the evolution of household wealth. Here, the contrast between initial high- and low savings is striking. In the first case (Figure 6a), although the savings rate declines, households continue to accumulate wealth, albeit at a reduced rate. In the low savings case (Figure 6b, which is the original Davies and Thurlow SAM), savings are insufficient to even maintain the initial level of wealth.

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[17] Relative to the BAU solution, real savings fall by 11.6% in the LoSH case, and by 13.2% in the HiSH case, while real consumption expenditures fall by 2.6% and 2.2%. It should be kept in mind, however, that real household consumption expenditures are computed here using the consumer price index, while real savings, from the point of view of capital accumulation, are based on the price index of the capital good. And in this particular simulation, under the LoSH case, the shock brings about a 7.4% fall in the price of capital, but a 9.5% drop in consumer prices relative to BAU; under the HiSH case, the reductions are 7.4% and 9.4%. Therefore, the discrepancy in the proportional reduction of savings and consumption is even greater in nominal than in real terms. In the LoSH case, there is a 55.1% fall in nominal savings, and a 11.8% fall in nominal consumption expenditures; the corresponding figures in the HiSH case are 19.6% and 11.4%.

Figure 6a

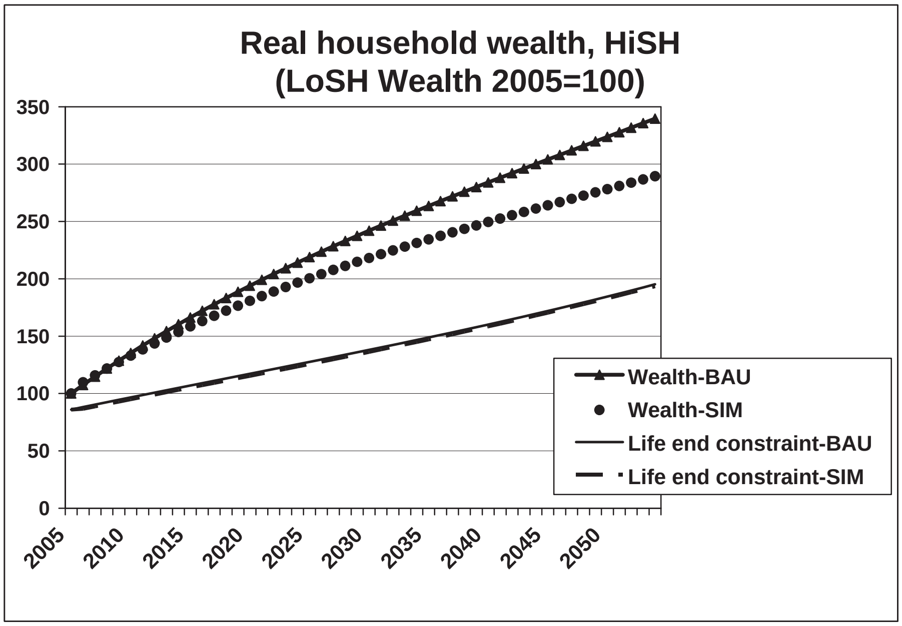
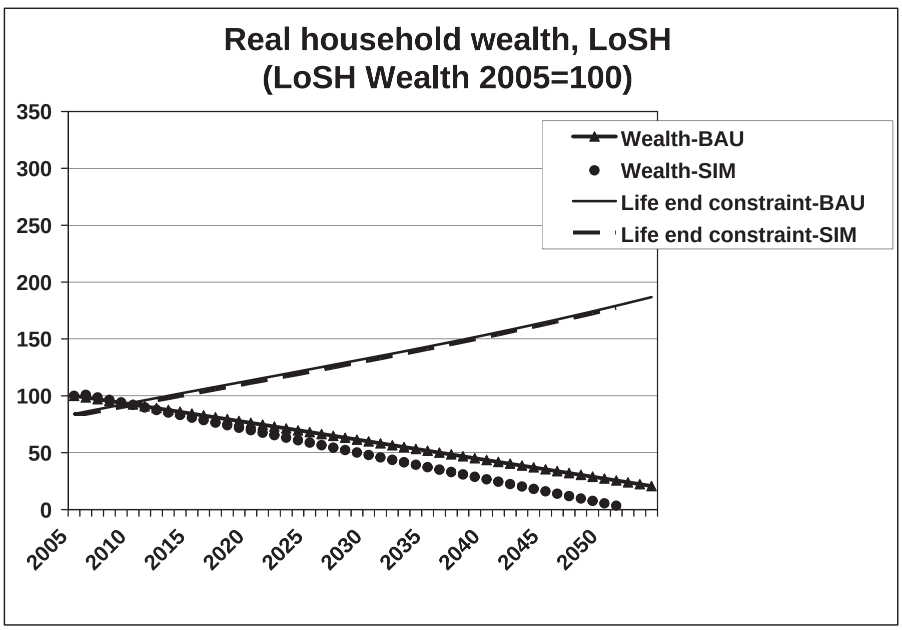


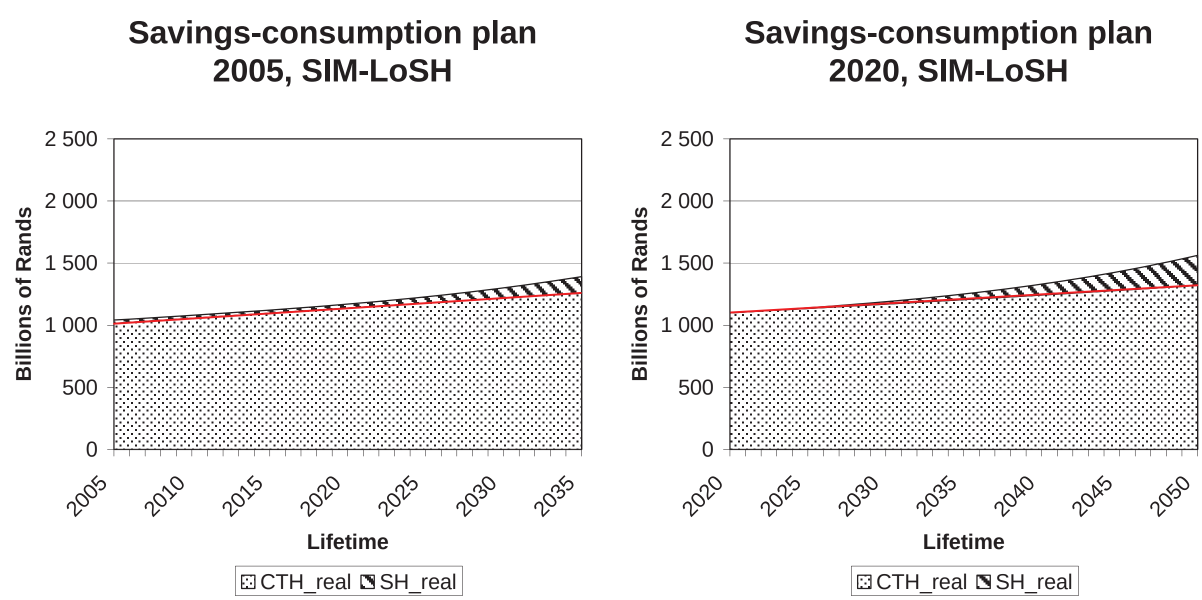
Figure 6b



Figures 6a and 6b also display the evolution of terminal wealth (Life end constraint), which defines the transversality (no Ponzi game) condition[18]. The terminal wealth constraint evolves according to the assumption that the household wants terminal wealth *per capita* to be equal to its initial wealth in real terms (equation M 120). And, as a matter of fact, the terminal wealth constraints are virtually identical across scenarios. Initially, real wealth is greater than its target value, because the target value calculation takes into account the expected evolution of consumer prices, which is negative (equation M 120). Both figures show a striking paradox: real wealth moves *away* from its target value! In the HiSH case, wealth is already greater initially than prescribed terminal wealth, and it keeps growing faster than the target is raised. In the LoSH case, wealth declines until it becomes less than terminal wealth, and it continues to fall as the target is raised. How is that possible?

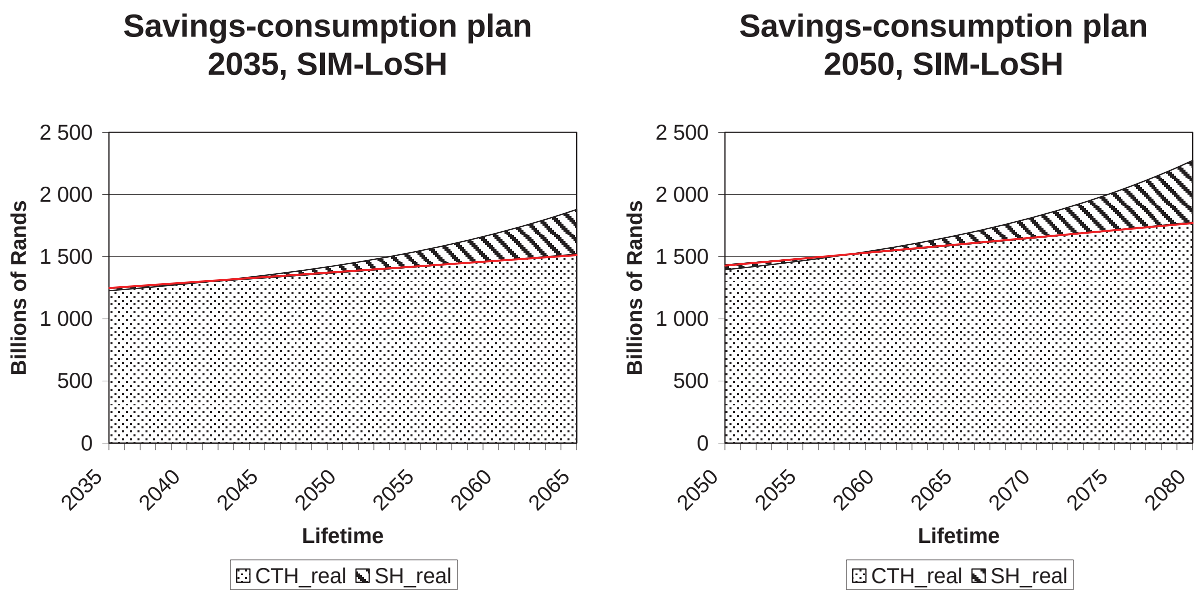
Figure 7 shows how, in the SIM-LoSH scenario, household savings and consumption plans change over the course of the 2005-2054 simulation. The first panel displays the savings and consumption plan as it stands in 2005 for the 2005-2035 period. It calls for small savings with a modest increase over time up to the planning horizon[19]. The three other panels present the thirty-year plans of 2020, 2035 and 2050

Figure 7



[18] Actually, real wealth and terminal wealth constraint are not really comparable because the latter should be discounted to its present value as it is in equation M 128.

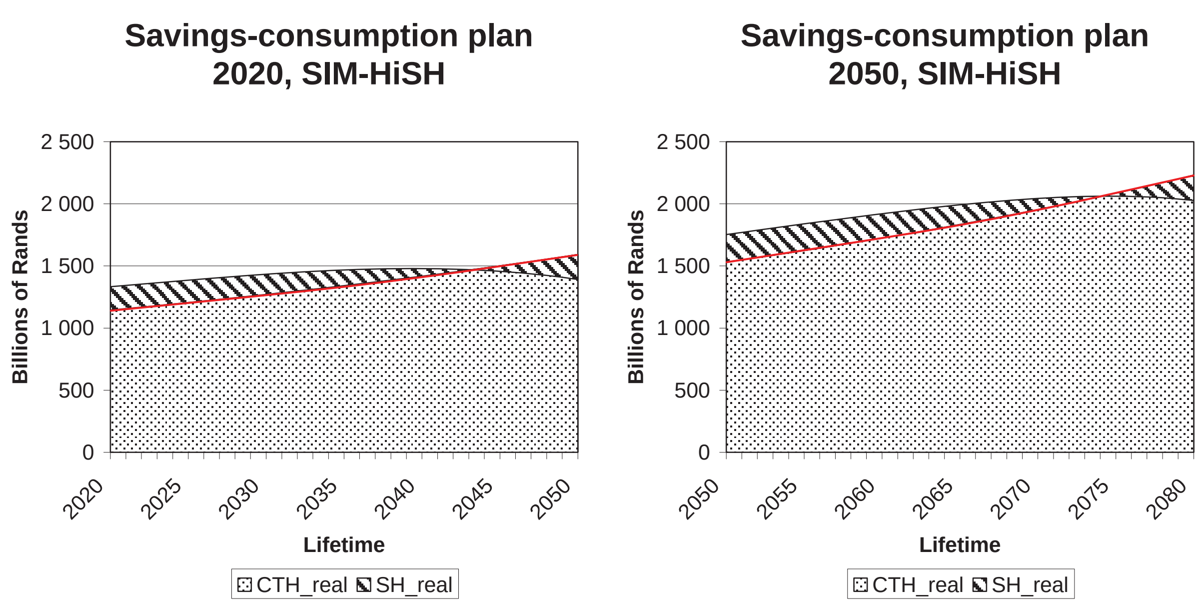
[19] The red line is planned consumption expenditures. The hatched area represents savings. When the hatched area is below the red expected-income line, as in the left part of the fourth panel (2050), savings are negative.



respectively. As time goes by, successive plans call for lower and lower savings in the current period (indeed, negative from 2017 onwards), with sharper planned increases in the future. Overall, good intentions never materialize, because only the first period of each successive plan is actually applied, and the model crashes in 2053 as household wealth threatens to turn negative.

Figure 8 displays household savings and consumption plans in the Sim-HiSH scenario, as they stand in 2020 and 2050. It can be seen that the situation is reversed compared to the LoSH situation. Households

Figure 8

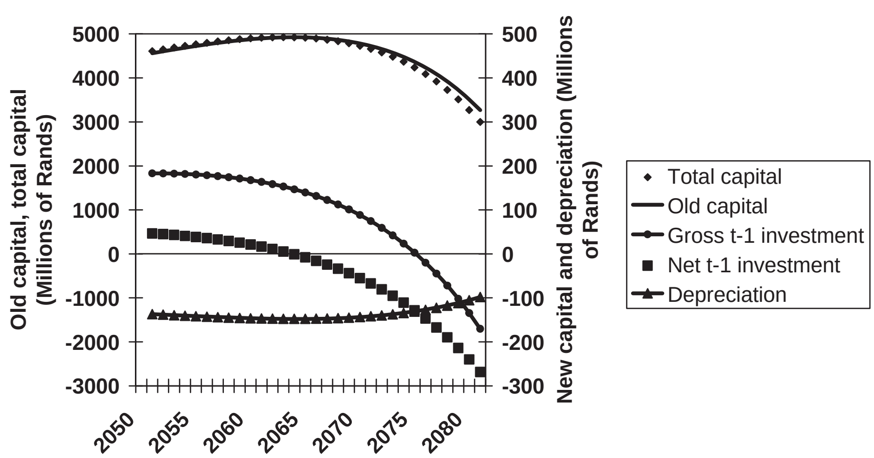


expect their income to peak at some point in the future, and then to decline. This foreseen evolution is the result of a slow anticipated increase in non-investment income (*YDHX*), combined with an evolution of capital income that reflects a planned wealth accumulation-decumulation cycle (see comments on Figure 9 below). Households plan to save and accumulate wealth for some time before dissaving, so that consumption expenditures can keep rising to the end of their planning horizon. As in the previous case, this pattern repeats itself in every period, and household wealth never ceases to accumulate.

Figure 9 details how, in the SIM-HiSH scenario, households' behavior in 2050 is consistent with expectations and plans, including the transversality condition. Planned savings are positive nearly to the end of the planning time span, but from 2064 onwards (less than midway to the planning horizon), it is foreseen that the previous period's savings (Gross t–1 investments) will be insufficient to replace depreciated capital in the household's wealth, which will henceforth decline to attain its terminal value in year 2081, as imposed by the transversality condition. The same situation repeats itself in every period of the simulation.

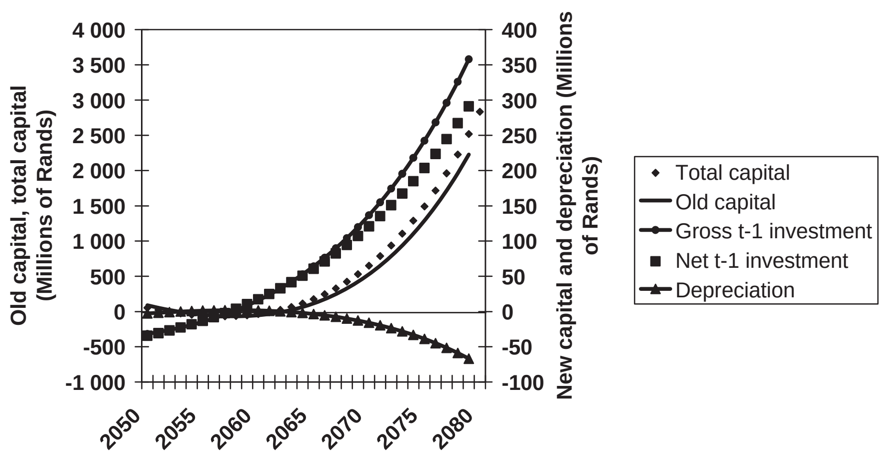
Figure 9

Expected evolution of wealth 2050, SIM-HiSH



Similar consistency is observed in the LoSH case, although the picture is a little murky. For the sake of completeness, Figure 10 details how, for example, in the SIM- LoSH scenario, household behavior in 2050 is consistent with expectations and plans, including the transversality condition.

Figure 10
Expected evolution of wealth 2050, SIM-LoSH



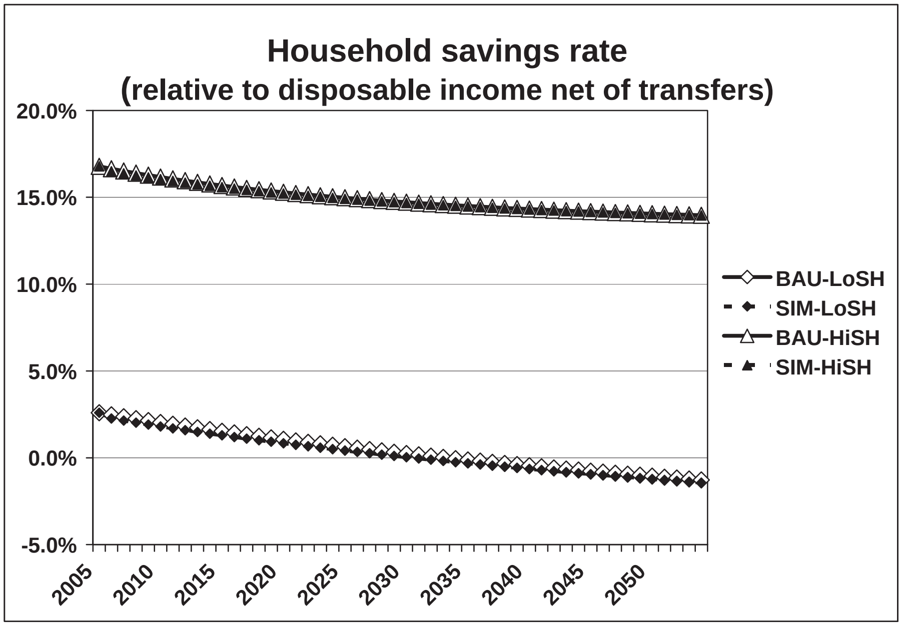
To summarize, household wealth drifts away from its target, even though the household consumption and savings decisions in each y ear, if their expectations for subsequent periods were fulfilled, would lead to an amount of wealth at the end of their t hirty-year planning time span that would be equal to the targeted amount (in other words, the intertemporal optimization solution has been verified to be correct).

3.5.2 Simulation 2: confiscation of 20% of household wealth

This simulation is an arti fice to see what the m odel predicts about household reaction to a sudden reduction in their wealth, while their end-of-life cons traint (transversality condition) remains unchanged. At the beginning of 2006 (y ear 2 of the sim ulation run), 20% of househol d ownership of capital is transfered to the government. This is achieved by an exogenous arbitrary 20% reduction in the household capital ownership share variables $\lambda^{RK}_{h,k,t}$, accompanied by a corresponding incre ase in government ownership shares. Thus, household wealth shrinks instantly by 20%, and so do household entitlem ents to capital income. Consequently, as households enter year 2006, they hold 20% less wealth in the simulation scenarios as in the BAU scenarios. We expected hous eholds to raise their savings rate in order to restore their level of wealth, and be able to attain their unchanged terminal wealth target.

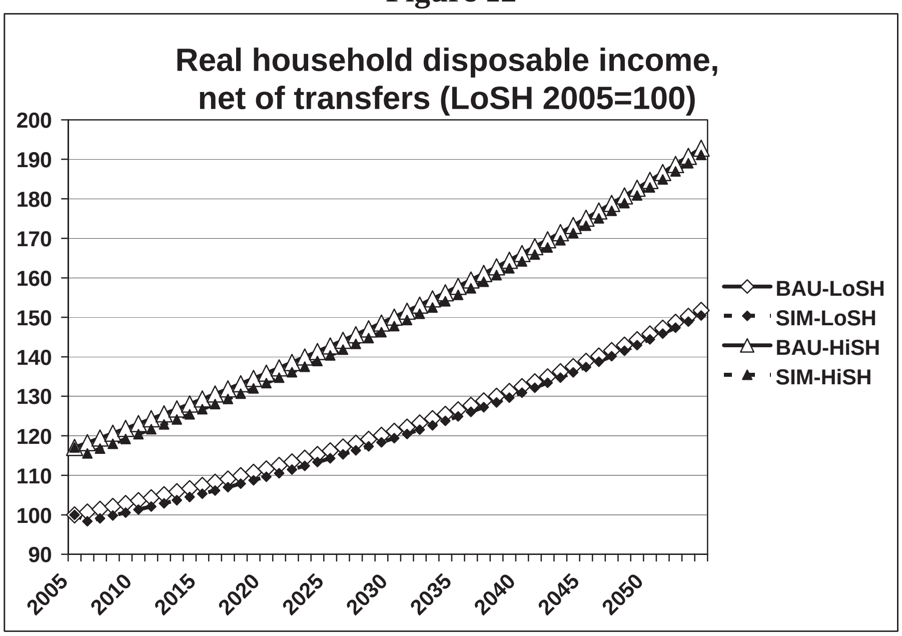
Figure 11 shows that, contrary to expectations, the household sa vings rate is virtually unaffected by the wealth shock.

Figure 11



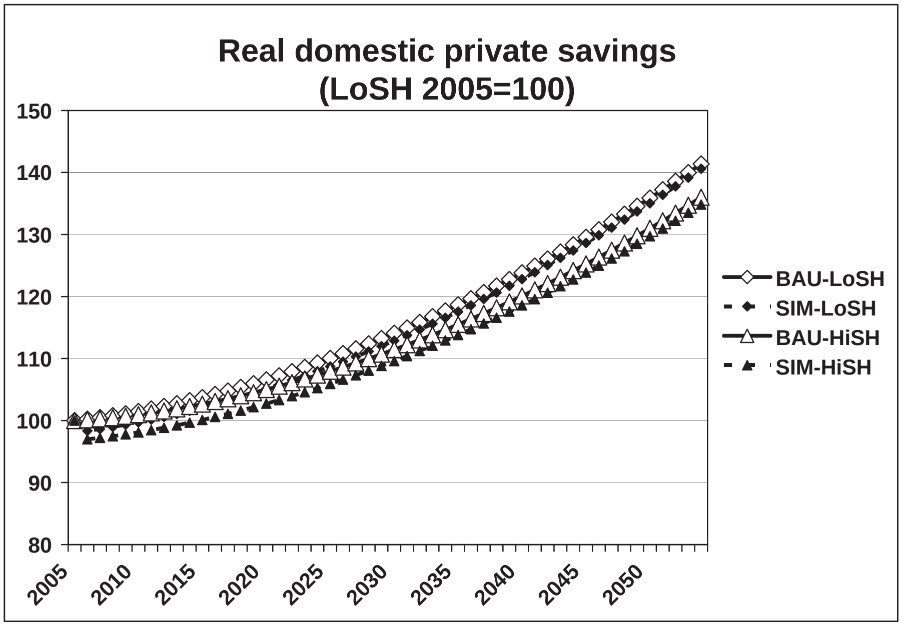
Of course, the shock does have some effect on real household disposable income, because of the loss of capital income (Figure 12).

Figure 12



As a result, real domestic private savings are somewhat dampened (Figure 13).

Figure 13



But this has little effect on household consumption, GDP and growth (through capital accumulation), and that effect fades away as time goes by (Figures 14 and 15).

Figure 14

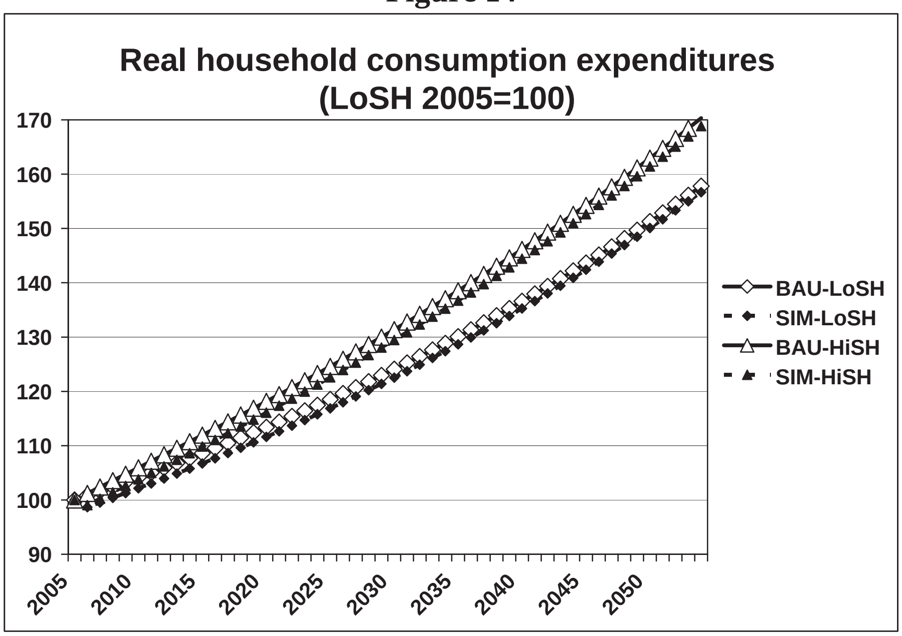
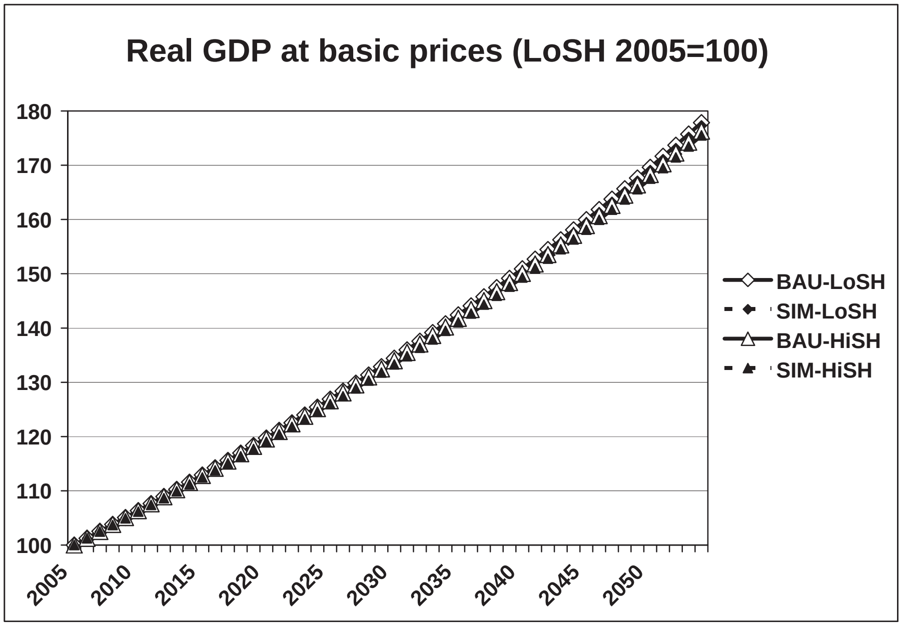


Figure 15



Why is the effect so weak? The explanation is found by examining the household consumption equation M 128, which is derived from intertemporal optimization.

M 128.	$CTH_{h,t} = \frac{1}{Z2_{h,t}}\left[\left(1 + RHO_{h,t} - \delta\right)KW_{h,t} + Z1_{h,t}YDHX_{h,t} - \frac{D_{h,LifeEnd,t}}{g_t^{PK\_PRI}}KW_{h,t}^{TERM}\right]$

Table 2 below shows the weight of each of the three components between square brackets in equation M 128, in the initial year 2005, and in 2006, after the shock in the SIM scenario, for both variants of the SAM.

Table 2 – Weights of the components of household consumption

	$\left(1 + RHO_{h,t} - \delta\right)KW_{h,t}$	$Z1_{h,t}YDHX_{h,t}$	$-\frac{D_{h,LifeEnd,t}}{g_t^{PK\_PRI}}KW_{h,t}^{TERM}$
LoSH			
BAU/SIM 2005	12.8%	89.4%	-2.2%
BAU 2006	12.5%	89.7%	-2.2%
SIM 2006	10.2%	92.0%	-2.2%
HiSH			
BAU/SIM 2005	12.5%	89.5%	-2.0%
BAU 2006	13.3%	88.7%	-2.0%
SIM 2006	10.8%	91.2%	-2.0%

The second of the three components between square brackets is the present value of non-investment income. It represents around 90% of the total. The shock on household wealth affects only the first term, which initially represents about 13% of the total. The tail cannot wag the dog...

The impact of the shock on wealth is dampened for another, more idiosyncratic, reason. The way the shock is specified, the wealth taken away from households is transferred to government. Consequently, government income from capital increases. With fixed government savings, there results an increase in current expenditures which stimulates labor demand, pushes the wage rate upward, and increases household labor income, partly compensating for the loss of capital income.

3.5.3 Simulation 3: 2%/10% surtax on household capital income

In this simulation, equation M 135 is modified to add a surtax on income from capital.

M 136.	$RHO_{h,t} = \left(1 - ttdh1_{h,t} - ttdhk_{h,t}\right) RRK_{h,t}$

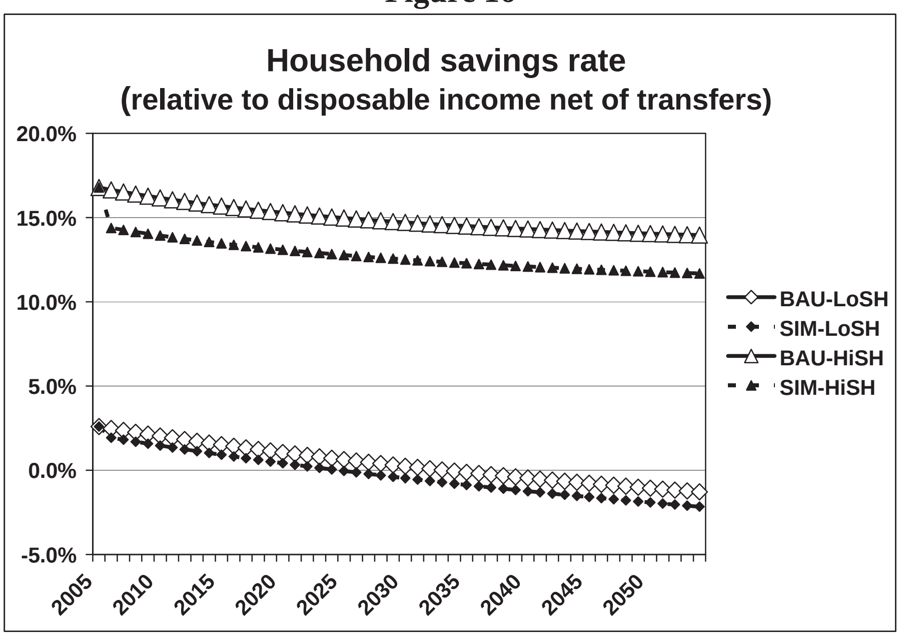
where

$ttdhk_{h,t}$: Rate of surtax on capital income of type *h* households

Equations M 111 and M 035 are modified accordingly. The surtax is 2% in the LoSH variant, and 10% in the HiSH variant. The modest 2% rate was chosen because a heftier rate made the model crash before 2054. This happened because, in the LoSH case, negative savings set on early in the simulation, and wealth threatened to turn negative.

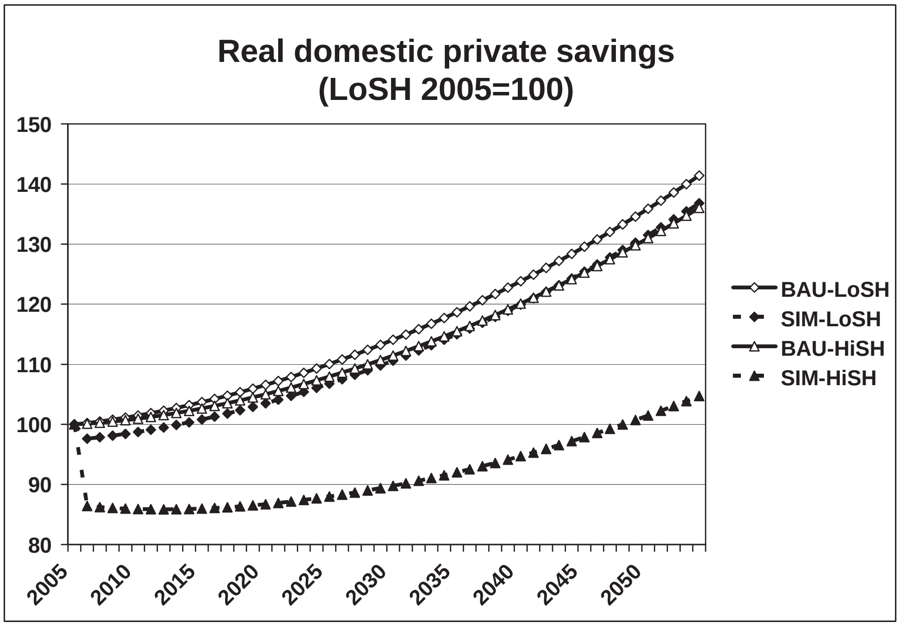
The surtax on capital income has a noticeable effect on the savings rate, as shown in Figure 16, compared to Figure 11.

Figure 16



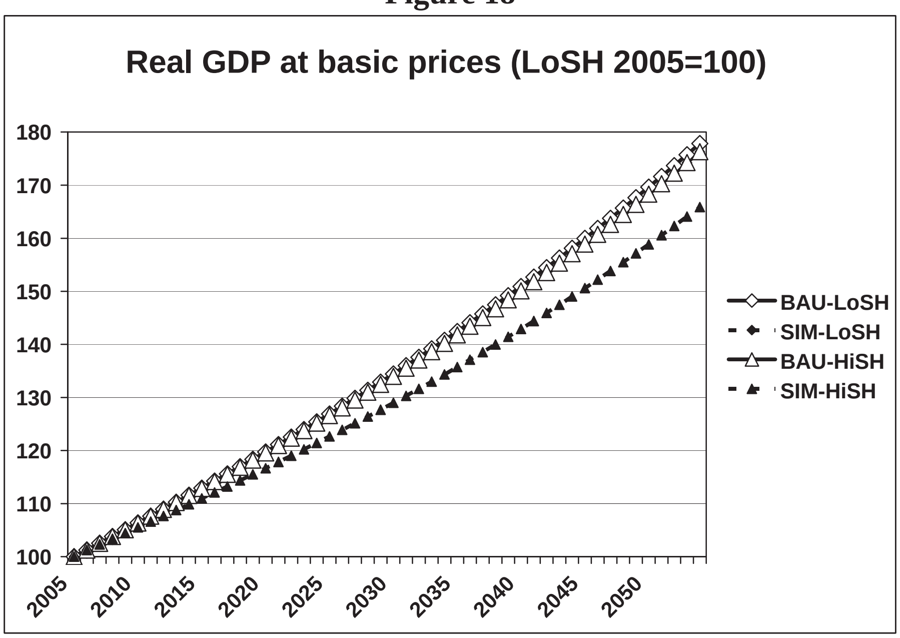
The reduction in the savings rate is reflected in the amount of domestic savings, as displayed in Figure 17. The impact is weaker in the LoSH case, because household savings represent only a small share of domestic savings (13.2% in 2005), whereas they are 100% of domestic savings in the HiSH case (by construction; see 3.1 above). The contrast with simulation 2 (Figure 13) is striking.

Figure 17



Lower savings mean less investment and slower capital accumulation. In the long run, this implies less growth and a lower real GDP, especially in the HiSH case, as shown in Figure 18.

Figure 18



3.5.4 Summary of simulation results

The first simulation, a permanent 50% drop in the international price of minerals, showed that the behavior predicted by "truncated rational expectations" is quite radically different from the one predicted by full rational expectations and perfect foresight. Indeed, in the savings-consumption plan that housesholds make every year, the future never materializes: household wealth keeps drifting away from its terminal constraint value (transversality condition), and that happens in spite of the fact that, if household expectations for subsequent periods were fulfilled, every year's plan would lead, thirty years down the road, to an amount of terminal wealth equal to the target.

The second simulation, a shock on the stock of wealth owned by households, has virtually no effect on household savings behavior. Our diagnostic is that the weight of wealth in consumption equation M 128 is insufficient for exogenous variations in wealth to have a strong impact on the consumption-savings balance.

The third simulation consisted in a (fiscal) shock on the rate of return to wealth. The results highlighted the role of the return rate as an incentive to save.

3.6 Alternative models

The same set of simulations were run with two other models based on the same South Africa SAMs.

3.6.1 Static expectations

In PEP-TRE, households apply intertemporal optimization in every period, based upon near-perfect foresight of the following period and extrapolation into the future. With static expectations, households expect the values of the relevant variables to remain at their current levels indefinitely. In our static-expectations model, households apply intertemporal optimization in every period based upon such static expectations. There is no need for recalibration as in PEP-TRE, and the GAMS coding of the model may be a lot simpler. Technically, however, we used the PEP-TRE code, skipping recalibration and fixing all growth factors exogenously at 1 (no growth).

Simulation 1: permanent 50% drop in the international price of minerals

We shall spare the reader a detailed examination of the results with the static-expectations model. The key difference is that the savings rate with static expectations is higher, both in the LoSH and in the HiSH case, except for the first 20 years (2006-2021) of the SIM-LoSH scenario, as displayed in Figure 19.

Figure 19a

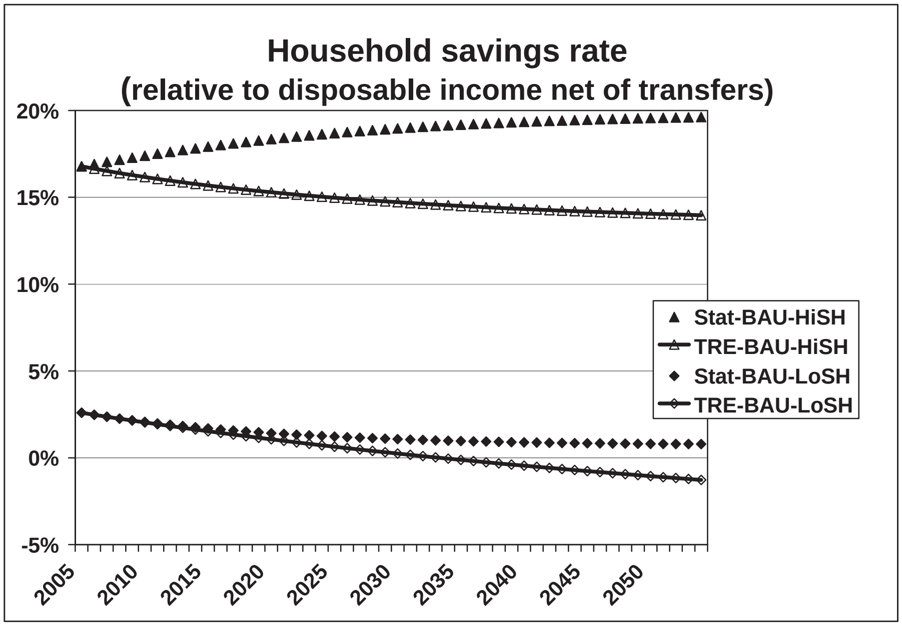
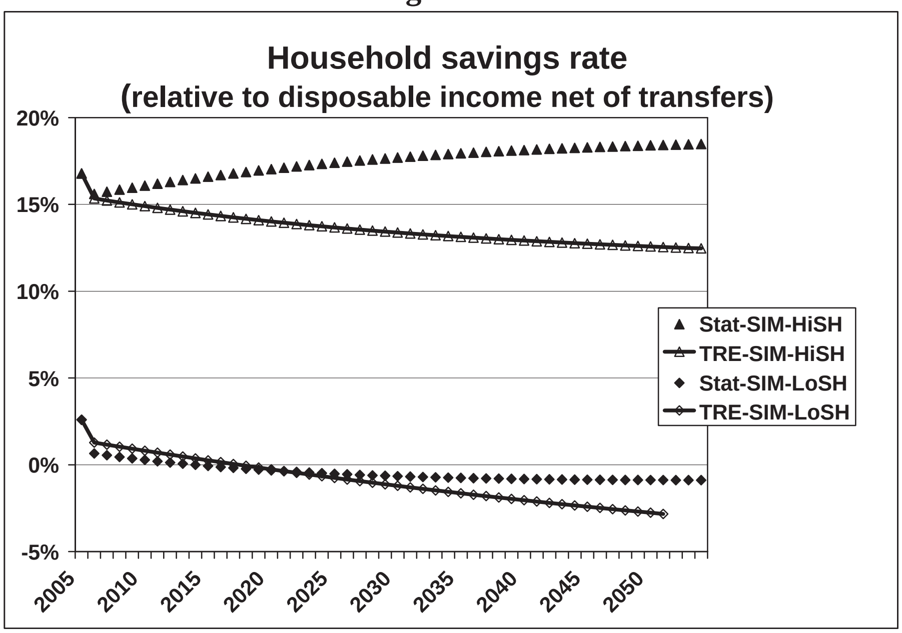


Figure 19b



It would appear that this is largely explained by the combined effects of the growth factors in PEP-TRE. In particular, the growth factors influence $Z1_{h,t}$ and $Z2_{h,t}$ directly, and indirectly through $D_{h,Lifetime,t}$ (equations M 125-127). We observe that the $Z1_{h,t} / Z2_{h,t}$ ratio is generally higher in the PEP-TRE model compared to the static-expectations model, which, from equation M 128, implies that households allocate a larger share of their non-investment income to consumption. Given the preponderance of non-investment income in determining the level of consumption expenditures (see Table 2), it follows that a higher $Z1_{h,t} / Z2_{h,t}$ ratio is likely to result in a lower savings rate.

Other differences between the static-expectations model and PEP-TRE follow. Higher savings rates result in larger domestic savings, more investment and capital accumulation, and accelerated growth.

Simulations 2 and 3

Just as with the PEP-TRE model, the 20% wealth meltdown has practically no effect. And the 2%/10% tax on capital income has a similar disincentive effect on savings, except that, as Simulation 1 leads us to expect, savings tend to be higher with the static-expectations model, especially in the HiSH variant.

3.6.2 Fixed savings rate

The same set of simulations were run for comparison purposes with a fixed-savings rate model. The model used is a modified version of PEP-1-t, where investment income shares evolve according to agents' savings and contribution to investment. In addition, the interest rate was redefined following equation M 129 above, and the scale parameter ϕ_t is treated as an endogenous variable, as explained in 2.2 and in the appendix.

Simulation 1: permanent 50% drop in the international price of minerals

The first obvious thing to observe is that the savings rate is greater in the fixed-savings rate model than in PEP-TRE (see Figure 5), since they fall in the latter ; compared to the static-expectations model, the savings rate is greater in the LoSH case, and less in the HiSH case (see Figure 19). It follows that capital accumulation and growth are higher with fixed household savings rates than in the two other models, except in the HiSH case, where they are lower than in the static-expectations model. The same is true of household wealth.

With fixed savings rates, the allocation of nominal disposable income after transfers between consumption and savings is constant. However, in simulations, consumer prices fall more than the price of the capital good relative to the BAU, so that in real terms, consumption expenditures contract less than savings.

Simulations 2 and 3

Just as with the PEP-TRE and static-expectations models, the 20% wealth meltdown has practically no effect. Finally, given the fixity of savings rates, the 2%/10% tax on capital income has no disincentive effect on savings, in sharp contrast with the two other models.

Conclusion: What remains to be done

Much remains to be explored and understood in these results. In particular, why do savings rates fall continuously in PEP-TRE, both in the high- and in the low savings rate situations? A conjecture is that this is related to the fact that prices tend to fall slowly in all scenarios, but the fall is less and less pronounced. Thus household price expectations, based on the current period and the next, overestimate the long term trend in price decline, and this leads them to under-save.

More generally, other simulations should be run, to more fully assess the responsiveness of household savings rates to changes in the rate of return on assets and to shocks on the stock of wealth. And the model's sensitivity to the arbitrary values of free parameters should be explored (Planning horizon, intertemporal elasticity of substitution, psychological discount rate, uniform rental rate of capital, specification of the terminal wealth constraint...).

Most importantly, the model presented here is incomplete in that, in its present state, it could not accommodate a SAM where some agents have negative savings. That is the reason why the 2005 South African SAM was chosen: all agents have positive savings. Indeed, we have assumed that ownership of the new capital created from investment is distributed in proportion to each agent's savings. But if some agents have negative savings, they draw from the pool of savings to equilibrate their budget. And so part of the savings of other agents is diverted from investment in productive capital. To correctly account for wealth accumulation, it is necessary to introduce (at least) another asset.

For example, if a country runs a current account surplus, then foreign savings are negative and there is an implicit capital-and-financial account flow of funds out of domestic savings to the RoW. It follows that domestic agents accumulate wealth partly in the form of investment abroad (portfolio investment or FDI). Similarly, if there is a government deficit, other agents accumulate wealth partly in the form of government debt securities (bonds). These other forms of wealth need to be taken into account if intertemporal optimization is to be consistent.

That's how Pandora's Box yawns wide open... Because accounting for financial assets poses several challenges. The first is a data challenge: we would need to have or construct data relating to stocks and flows of financial assets, and to flows of income paid and received in relation to financial assets. The second is a modelling challenge: if savings are to be explicitly allocated to various assets, then we need a portfolio allocation mechanism and a pricing mechanism. Similar challenges arise in the context of world models which take into account capital-and-financial account flows and net international investment positions (Lemelin *et al.* 2013), and perhaps some of the solutions developed in that context can be transposed to PEP-TRE.

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Appendix : Aggregation

Activities and Commodities

	Original classification		Aggregation	
	Code	**Description**	**Code**	**Description**
1	agri	Agriculture	Agr-for-fsh	Agriculture, forestry & fisheries
2	fore	Forestry	Agr-for-fsh	Agriculture, forestry & fisheries
3	fish	Fisheries	Agr-for-fsh	Agriculture, forestry & fisheries
4	coal	Coal mining	Mining	Mining
5	omin	Other mining	Mining	Mining
6	meat	Meat	Food	Food
7	pfsh	Fish	Food	Food
8	fveg	Fruit & vegetables	Food	Food
9	oils	Oils & fats	Food	Food
10	dair	Dairy	Food	Food
11	mill	Grain milling	Food	Food
12	star	Starches	Food	Food
13	feed	Animal feeds	Food	Food
14	bake	Bakery	Food	Food
15	sugr	Sugar	Food	Food
16	conf	Confectionary products	Food	Food
17	past	Pastas	Food	Food
18	food	Other foods	Food	Food
19	btob	Beverages & tobacco	Bev&tob	Beverages & tobacco
20	fabr	Weaving & finishing of fabrics	Tex&cloth	Textiles & clothing
21	made	Made-up textiles	Tex&cloth	Textiles & clothing
22	carp	Carpets, rugs & mats	Tex&cloth	Textiles & clothing
23	text	Other textiles	Tex&cloth	Textiles & clothing
24	knit	Knitting & crocheted fabrics	Tex&cloth	Textiles & clothing
25	wear	Wearing apparel	Tex&cloth	Textiles & clothing
26	leat	Leather products	Tex&cloth	Textiles & clothing
27	foot	Footwear	Tex&cloth	Textiles & clothing
28	wood	Wood products	Wood&pap	Wood & paper
29	papr	Paper products	Wood&pap	Wood & paper
30	prnt	Printing & publishing	Wood&pap	Wood & paper
31	petr	Petroleum products	Petro	Petroleum products
32	bchm	Basic chemicals	Chemicals	Chemicals
33	fert	Fertilizers & pesticides	Chemicals	Chemicals
34	pain	Paints & related products	Chemicals	Chemicals
35	phar	Pharmaceuticals	Chemicals	Chemicals
36	soap	Soap & related products	Chemicals	Chemicals
37	ochm	Other chemicals	Chemicals	Chemicals
38	tyre	Rubber tyres	Chemicals	Chemicals
39	rubb	Other rubber products	Chemicals	Chemicals
40	plas	Plastics	Chemicals	Chemicals
41	glas	Glass products	Non-metall	Non-metallic minerals
42	cere	Ceramicware	Non-metall	Non-metallic minerals
43	ceme	Cement	Non-metall	Non-metallic minerals

Activities and Commodities

	Original classification		Aggregation	
	Code	Description	Code	Description
44	nmet	Other non-metallic minerals	Non-metall	Non-metallic minerals
45	iron	Basic iron & steel	Metal	Metal products
46	nfer	Non-ferrous metal	Metal	Metal products
47	metp	Metal products	Metal	Metal products
48	engn	Engines & turbines	Mechanic	Mechanical equipment
49	pump	Pumps, compressors & valves	Mechanic	Mechanical equipment
50	bear	Bearings & gears	Mechanic	Mechanical equipment
51	lift	Lifting equipment	Mechanic	Mechanical equipment
52	gmch	General purpose machinery	Mechanic	Mechanical equipment
53	smch	Special purpose machinery	Mechanic	Mechanical equipment
54	appl	Domestic appliances	Elec&Eltron	Electric & electronic
55	omch	Office machinery	Elec&Eltron	Electric & electronic
56	emch	Electrical machinery	Elec&Eltron	Electric & electronic
57	rtel	Radio & television equipment	Elec&Eltron	Electric & electronic
58	mequ	Medical equipment	Elec&Eltron	Electric & electronic
59	vehe	Vehicles & parts	Trnsp_eqp	Transport equipment
60	ship	Ships & boats	Trnsp_eqp	Transport equipment
61	rail	Railways & trams	Trnsp_eqp	Transport equipment
62	airc	Aircraft	Trnsp_eqp	Transport equipment
63	otrn	Other transport equipment	Trnsp_eqp	Transport equipment
64	furn	Furniture	Oth_manu	Other manufacturing
65	jewe	Jewellery	Oth_manu	Other manufacturing
66	oman	Other manufacturing	Oth_manu	Other manufacturing
67	rcyc	Recycling & waste	Oth_manu	Other manufacturing
68	elec	Electricity & gas distribution	E&Gdistrib	Electricity & gas distribution
69	watr	Water distribution	WatDistrib	Water distribution
70	cons	Construction	Construc	Construction
71	trad	Wholesale & retail trade	Trade	Wholesale & retail trade
72	hotl	Hotels & catering	Hotels	Hotels & catering
73	tran	Transport	Transport	Transport
74	comm	Post & communications	Post&comm	Post & communications
75	fsrv	Financial services	Fin_serv	Financial services
76	insu	Insurance & pensions	Ins&pens	Insurance & pensions
77	real	Real estate activities	Real_est	Real estate activities
78	rdev	Research & development	Oth_bus	Other business services
79	legl	Legal & accounting activities	Oth_bus	Other business services
80	rent	Rental services	Oth_bus	Other business services
81	busi	Other business activities	Oth_bus	Other business services
82	govn	Public administration	Pub_adm	Public administration
83	educ	Education	Education	Education
84	heal	Health	Health	Health
85	osrv	Other services	Oth_serv	Other services

Appendix : The endogenous investment scale variable as a rationing device

In PEP-1-t, the investment equation is

108. $$IND_{k,bus,t} = \phi_{k,bus}\left[\frac{R_{k,bus,t}}{U_{k,bus,t}}\right]^{\sigma^{INV}_{k,bus}} KD_{k,bus,t}$$

with

109. $$U_{k,bus,t} = PK_t^{PRI}\left(\delta_{k,bus} + IR_t\right) \text{ and } U_{k,pub,t} = PK_t^{PUB}\left(\delta_{k,pub} + IR_t\right)$$

In practice however, the PEP-1-t calibration procedure results in uniform values for $\phi_{k,j}$, so we might as well write PEP-1-t equation 108 as

iii . $$IND_{k,bus,t} = \phi^* \left[\frac{R_{k,bus,t}}{U_{k,bus,t}}\right]^{\sigma^{INV}_{k,bus}} KD_{k,bus,t}$$

The interest rate IR_t appears nowhere else in the PEP -1-t model, and it takes whatever value it m ust for the equilibrium constraint

105. $$IT_t^{PRI} = PK_t^{PRI} \sum_{k,bus} IND_{k,bus,t}$$

to be satisfied.

In PEP-TRE, however, the forward-looking interest rate is tied to the expected rate of return on capital :

M 129.	$IR_t = \left[1 + \frac{\sum_h \left[g_{h,t}^{RRK} RRK_{h,t} KW_{h,t}\right]}{\sum_h KW_{h,t}} - \delta\right] g_t^{PK\_PRI} - 1$

To clarify the relationship between the two formulations, let us rename the PEP-1-t « wild card » interest rate as IR_t^*, and the corresponding user cost of capital as

iii . $$U^*_{k,bus,t} = PK_t^{PRI}\left(\delta_{k,bus} + IR_t^*\right) \text{ and } U^*_{k,pub,t} = PK_t^{PUB}\left(\delta_{k,pub} + IR_t^*\right)$$

It is required that

iii . $$IND_{k,bus,t} = \phi^* \left[\frac{R_{k,bus,t}}{U^*_{k,bus,t}}\right]^{\sigma^{INV}_{k,bus}} KD_{k,bus,t} = \phi_t \left[\frac{R_{k,bus,t}}{U_{k,bus,t}}\right]^{\sigma^{INV}_{k,bus}} KD_{k,bus,t}$$

It follows that the following must hold

iii . $$\phi^* \left[\frac{1}{U^*_{k,bus,t}}\right]^{\sigma^{INV}_{k,bus}} = \phi_t \left[\frac{1}{U_{k,bus,t}}\right]^{\sigma^{INV}_{k,bus}}$$

Whence

iii . $$\phi_t = \phi^* \left[\frac{U_{k,bus,t}}{U^*_{k,bus,t}}\right]^{\sigma^{INV}_{k,bus}}$$

iii . $$\phi_t = \phi^* \left[\frac{PK^{PRI}_t\left(\delta_{k,bus} + IR_t\right)}{PK^{PRI}_t\left(\delta_{k,bus} + IR^*_t\right)}\right]^{\sigma^{INV}_{k,bus}}$$

iii . $$\phi_t = \phi^* \left[\frac{\delta_{k,bus} + IR_t}{\delta_{k,bus} + IR^*_t}\right]^{\sigma^{INV}_{k,bus}}$$

The endogenous scale variable ϕ_t is a function of the calibrated fixed scale parameter ϕ^* and of both the forward-looking interest rate and the PEP-1-t wild card rationing interest rate . An equivalent model formulation would be to maintain fixed investment scale parameters in PEP-TRE as in PEP-1-t, and define two interest rates, a forward-looking rate of interest that guides households in their intertemporal optimization, and a wild card cost-of-borrowing rate of interest that rations investible funds among competing uses. But what, then, would be the relationship between the two? That question is hard to answer, given that the Jung-Thorbecke-inspired investment equation is only loosely related to Tobin's *q*-theory of investment, as discussed in Part 1 of Lemelin and Decaluwé (2007, especially p. 29-30).

Finally, it could be mentioned somewhat crassly that the MIRAGE model uses an endogenous scale variable similar to ϕ_t to ration investment.