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Monitoring for Disruptions in Financial Markets

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Monitoring for Disruptions
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Résumé / *Abstract*

Nous étudions les tests CUSUM historiques et séquentiels pour des séries dépendantes avec des applications en finance. Pour les processus temporels, une nouvelle dimension se présente : l'effet du choix de la fréquence des observations. Un nouveau test est également proposé. Ce test est basé sur une formule de prévision locale d'un pont brownien.

Mots clés : changement structurel, CUSUM, GARCH, variation quadratique, 'power variation', données de haute fréquence, pont Brownien, puissance locale, tests séquentiels.

Historical and sequential CUSUM change-point tests for strongly dependent nonlinear processes are studied. These tests are used to monitor the conditional variance of asset returns and to provide early information regarding instabilities or disruptions in financial risk. Data-driven monitoring schemes are investigated. Since the processes are strongly dependent several novel issues require special attention. One such issue is the sampling frequency. We study the power of detection as sampling frequencies vary. Analytical local power results are obtained for historical CUSUM tests and simulation evidence is presented for sequential tests. Finally, a prediction-based statistic is introduced that reduces the detection delay considerably. The prediction based formula is based on a local Brownian bridge approximation argument and provides an assessment of the likelihood of change-points.

Keywords: structural change, CUSUM, GARCH, quadratic variation, power variation, high frequency data, Brownian bridge, boundary crossing, sequential tests, local power.

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1 Introduction

One of Clive Granger's very first papers was on the prediction of extreme events, namely floods of tidal rivers (see Granger (1959)). Since Granger's early work there has been an enormous growth in the area of modelling extreme events in economics and finance (see e.g. Embrechts et al. (1997)). Despite the significant progress we made, there are still many outstanding issues. This paper is about monitoring for instabilities or disruptions in financial markets and the use of sequential testing procedures.[1] Structural breaks and extreme tail observations are two types of rare events. The latter has been the main focus of risk management whereas the former represents a fundamental shift in the distribution of risky outcomes. Testing for breaks has obviously also close connections with model specification analysis as well as forecasting. Out-of-sample prediction is typically based on the maintained assumption of model (parameter) stability (see e.g. Hendry (1997)). Sequential analysis is therefore a desirable tool when out-of-sample analysis is performed since it allows for real-time monitoring of prediction models. Moreover, monitoring financial risk can be considered as a useful statistical tool for financial crises.

So far, most statistical and econometric methods of sequential change-point analysis have focused on linear regression models or linear dynamic models with weak dependence. Examples include Lai (1995, 2001), Chu et al. (1996), Leisch et al. (2000), Zeileis et al. (2004), among others. Notable exceptions are Altissimo and Corradi (2003) and Berkes et al. (2004) that will be discussed shortly. In this paper we are interested in monitoring stability of nonlinear strongly dependent processes. The interest stems from the fact that a large area of important sequential analysis applications is financial risk management. The daily reporting of some risk exposure measure has been adopted by many financial regulators. Embedded in most measures is a prediction rule based on a model for portfolio returns. For example, Value-at-Risk (VaR) attempts to forecast likely future losses using quantiles of the (conditional) portfolio return distribution and Expected Shortfall (ES) measures the expected loss given it exceeds some threshold. Andreou and Ghysels (2003) showed that ignoring structural change will lead to financial losses beyond those that can be anticipated by the routine application of

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[1]We will use the terms change-point and disruptions interchangeably. Hence, disruptions in financial markets are structural breaks, or change-points in the asset returns process.

risk management estimation or simulation methods. The idea to monitor sequentially asset return related processes for change-points is still relatively unexplored and raises many issues. We focus exclusively on CUSUM-type tests and provide a number of contributions towards a better understanding of sequential testing for structural breaks in financial asset return processes or more generally nonlinear dynamic strongly dependent processes. In this respect our paper fits in a broader research program addressing the challenges of quality control for dependent monitoring processes (see e.g. Lai (2001, p. 317-8) and Stoumbos et al. (2000, p.994-5, sections 6 and 8)).

The first issue we emphasize is the use of data-driven monitoring processes. Most of the literature so far has emphasized sequential testing via monitoring (individual) coefficients of parametric models. One of the main innovations in Andreou and Ghysels (2003) is the use of empirical monitoring processes that are purely data-driven and based on high frequency financial time series processes. Indeed, one virtue of financial market applications is the availability of high frequency price data for actively traded assets that bear neither sampling costs nor measurement error. These easy to collect and compute processes are practically appealing as even for the simplest representative parametric model for asset returns, e.g. a GARCH process, the likelihood function may be extremely complicated (for cases other than the Normal or Student's t distribution) and therefore make parametric sequential analysis computational intensive and cumbersome. We do not necessarily advocate the exclusive use of data-driven tests, as parametric model-based tests can provide complementary information. For instance, Berkes et al. (2004) derive a sequential test for the parameters of a GARCH sequence. Their test could be used alongside the procedures presented here since they can provide information regarding the type of the change-point (e.g. constant versus dynamics of volatility). On the other hand Altissimo and Corradi (2003) deal with sequential tests for multiple breaks in the mean instead of the volatility of α-mixing processes.[2]

The second issue we explore is the effect of sampling frequency on the local power of CUSUM tests. Since we treat dependent processes, the choice of sampling frequency can be critical, an issue that has not been addressed in the literature since most of the emphasis has been on monitoring i.i.d. processes. Sampling frequency is indeed often one of the choice variables in the context of financial time series applications. We tackle the local asymptotic power

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[2] Andreou and Ghysels (2002) also consider multiple change-point tests in volatility.

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analytically in the context of a historical or off-line CUSUM statistic. More specifically, we find that the local power of the CUSUM-type ARCH tests for stability are inextricably linked with the sampling frequency choice and temporal aggregation involved in volatility models. This yields some interesting power trade-offs between the persistence and kurtosis properties of the process for different sampling frequencies. We focus exclusively on the ARCH(1) process where persistence and kurtosis are described each by a single parameter and quantify the trade-off between tail behavior, persistence, local power and sampling frequency. However, the analytical complexity of sequential CUSUM testing precludes us from deriving closed form solutions for the local power of tests with varying sampling frequency. Hence we resort to a comprehensive simulation study in order to examine the properties of on-line change-point tests and show that the same trade-offs appear in sequential analysis.

The third and final contribution pertains to a new prediction-based formula that has at least two appealing features. It uses a local Brownian bridge approximation argument and is shown to (1) substantially reduce the delay in detection time of structural breaks and (2) provide a probability statement about the likelihood of the occurrence of a structural break.

The remainder of the paper is organized as follows. In section 2 we present the test statistics and empirical processes to which they apply. Section 3 covers asymptotic local power with varying sampling frequencies and the next section 4 deals with sequential CUSUM tests. Section 5 reports an extensive simulation experiment on the various aspects of sequential CUSUM test applications in monitoring financial asset returns. Section 6 introduces the prediction-based formula for monitoring processes. Section 7 concludes the paper.

2   Data, Models and Test Statistics

We start from standard conditions in asset pricing theory. In particular, absence of arbitrage conditions and some mild regularity conditions imply that the log price of an asset must be a semi-martingale process (see e.g. Back (1991) for further details). Applying standard martingale theory, asset returns can then be uniquely decomposed into a predictable finite variation component, an infinite local martingale component and finally a compensated jump martingale component. This decomposition applies to both dis-

crete and continuous time settings and is based on the usual filtration of past returns. The predictable component is the expected return, i.e. is the conditional mean of the process given past returns. Most efforts in financial modelling have focused on two aspects: (1) the conditional volatility dynamics given that returns exhibit non-linear dependence and (2) the conditional tail behavior that pertains to the process of returns normalized by a conditional volatility measure, i.e. r_t/σ_t. For ease of presentation let us consider a discrete time setting and the general location-scale nonlinear time series model for the returns process $\{r_t\}$:

$$r_t = \mu(\mathbf{r}_{t-1}^0, \beta_1) + \sigma(\mathbf{r}_{t-1}^0, \beta_2)u_t, \quad t \geq 1 \qquad (2.1)$$

where the errors $\{u_t, t \geq 1\}$ are independent of $\mathbf{r}_{t-1}^0 := (r_{t-1}, r_{t-2}, ...)'$ and i.i.d. with distribution function $f(u, \beta_3)$. The known functions $\mu \rightarrow \mathbf{R}$ and $\sigma \rightarrow \mathbf{R}^+$ have unknown parameters β_1 and β_2, respectively. A parametric approach involves explicit parameterizations of $\mu_t \equiv \mu(., \beta_1)$, $\sigma_t \equiv \sigma(., \beta_2)$ and usually also the distribution function $f(u, \beta_3)$ in (2.1). Parametric models within the class of ARCH and SV models can be nested into equation (2.1). The former class of models assumes that σ_t is a measurable function of past returns, whereas the latter assumes that volatility is latent (see Bollerslev et al. (1994) and Ghysels et al. (1996) for further discussions). It should be noted that the specification in (2.1) amounts to working with the location scale family of distributions for the conditional distributions once the conditional mean and variance are determined.

Equation (2.1) represents a class of nonlinear time series models where the structure of the conditional mean and variance needs to be a priori specified. In a standard parametric or a semi-parametric setting, misspecification of the conditional mean and/or variance may invalidate consistent estimation. The parametric approach has certain advantages as to identifying the sources of structural breaks, yet is quite restrictive for financial asset returns due to the plethora of alternative stylized facts and models suggested for the conditional variance specification (e.g. the large family of ARCH- and SV-type models) and for the distribution function. Moreover, the computational burden of repeated semi-parametric estimation may be prohibitively expensive on a daily basis. The nonlinear optimization involved in sequentially estimating the GARCH parameters requires large samples (1000 or more daily observations). Additional estimation costs involved in the Berkes et al. (2004) approach involve the matrix of the product of the conditional pseudo-likelihood and its partial derivative which does not possess a closed-form

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formula and for which the exact GARCH parameters are used. In addition, the high frequency (intradaily) data exploited in our empirical monitoring processes yield more accurate estimates of volatility than parametric models do based on lower frequency (daily) data. Similarly the data-driven monitoring processes involving high frequency data can be computed at virtually any sampling frequency, that is daily, half-daily, hourly, etc. This is almost impossible for parametric and semi-parametric settings as finer sampling greatly increases the complexity of the models involved. This feature will be extremely useful as we will be able to address the issue of sampling frequency and the properties of sequential analysis in a feasible practical context. We describe in a first subsection on the monitoring processes. A second subsection covers CUSUM statistics.

2.1   Empirical monitoring processes

For monitoring financial risk one is interested typically in daily and monthly horizons. At such sampling frequencies returns have negligible drift terms, i.e. predictable component, as argued by Merton (1980), so that we will proceed with μ_t equal to zero. We present various empirical processes that can be used for monitoring purpose in a sequential analysis setting. In the context of Risk Management Quality Control (RMQC) Andreou and Ghysels (2003) distinguish three classes of empirical monitoring processes. The first class they consider involves observed (possibly high frequency) return processes and transformations of raw returns.[3]

Returns sampled at a daily frequency will be denoted r_t. For the purpose of estimating volatility we will also consider $r_{(m),t-j/m}$, the jth discretely observed time series of continuously compounded returns with m observations per day (with the index "t-j/m" referring to intra-daily observations). Hence, the unit interval for $r_{(1),t}$ is assumed to yield the daily return (with the subscript "(1)" typically omitted so that r_t will refer to daily returns).

The first monitoring process is daily squared returns. They are a measure of volatility, albeit a very noisy one, as discussed in detail by Andersen et al. (2003). A second functional transformation of returns is $|r_t|$, a process examined in detail by Ding et al. (1993) and shown to have much higher persistence and evidence of long memory than squared returns. It is often argued that

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[3] The second and third class of monitoring processes not considered here involve respectively returns standardized by some conditional variance filter and estimated parameter monitoring processes. See Andreou and Ghysels (2003) for further details.

in the presence of deviations from normality absolute values could be more robust than squared values for conditional variance estimation (see e.g. Davidian and Caroll (1987)). The idea that volatility can be precisely estimated using high frequency data goes back at least to Merton (1980) and has been the subject of recent research given the availability of high frequency financial data. The context is typically one of stochastic volatility diffusions with a jump component: $dr_u = \mu_u + \sigma_u dW_u + J_u d\lambda_u$, which discretize to equations such as (2.1). High frequency data volatility estimators computed as $\sum_{j=0}^{m-1} [r_{(m),t-j/m}]^2$, are denoted $\tilde{Q}^{(m)}_{[t-1,t]}$ as it involves a discretization based on m intradaily returns and pertains to the increments in the quadratic variation of day t, $\int_{t-1}^{t} r_u^2 du$. These processes have been studied extensively by Andersen et al. (2003), Barndorff-Nielsen and Shephard (2002), among others. To simplify notation we will refer to this third class of monitoring processes as QV_t since they refer to the daily increments in quadratic variation. It was noted that absolute returns are more robust to outliers, therefore we also consider as fourth monitoring process $\sum_{j=0}^{m-1} |r_{(m),t-j/m}|^p$, pertaining to the increments in the pth power variation of day t. In empirical work one often uses the so called realized power variation $\tilde{P}^{(m)}_{[t-1,t]}$ where $p = 1$, i.e. the cumulative sum of absolute intradaily returns is computed. We will henceforth denote this process as PV_t. A final monitoring process we will consider is the industry standard *Riskmetrics* volatility estimator since financial market practitioners make use of this measure. This daily volatility estimate will be denoted RM_t, and is an industry benchmark filter for volatility defined and an exponentially weighted moving average of squared daily returns, r_t^2, given by $RM_t = \lambda RM_{t-1} + (1 - \lambda)r_t^2$ where $\lambda = 0.94$ for daily data.

In the rest of the paper we will focus exclusively on the above monitoring processes, but our coverage will at times be selective since the results often show similarities across the different volatility filters.

2.2 Historical CUSUM tests for volatility models

First we consider the historical CUSUM tests as it allows us to derive the local asymptotic behavior under the alternative of a break with various sampling frequencies. Subsequently we address sequential CUSUM tests. The interest in applying historical CUSUM tests to volatility has been motivated by the observation of various authors, including Granger (1998), Lobato and Savin (1998), Granger and Hyung (1999), Mikosch and Starica (2004), among others, that the presence of breaks may explain the findings of long memory

in volatility. Andreou and Ghysels (2002) use historical CUSUM-type procedures to test for structural breaks in volatility. The object of interest is the volatility process σ_t, and we use the various data-driven volatility-related processes described in the previous subsection to monitor for breaks. To do so we first turn our attention to parametric model-based CUSUM tests, as they provide the background for our data-driven analysis. We provide first a brief discussion of the Kokoszka and Leipus (2000) where the process monitored for homogeneity is $|r_t|^{\delta}$, $\delta = 1, 2$.

In order to test for breaks in an ARCH(∞) Kokoszka and Leipus (2000) consider the following process:

$$U_T(k) = \left(1/\sqrt{T}\sum_{j=1}^{k} X_j - k/(T\sqrt{T})\sum_{j=1}^{T} X_j\right) \qquad (2.2)$$

where $0 < k < T$, $X_t = r_t^2$. The returns process $\{r_t\}$ follows an ARCH(∞) process, $r_t = u_t\sqrt{h_t}$, $h_t = b_0 + \sum_{j=1}^{\infty} b_j r_{t-j}^2$, $a \geq 0, b_j \geq 0, j = 1, 2, \ldots$ with finite fourth moment and errors that can be non-Gaussian. The CUSUM type estimator $\hat{k}$ of a change point k^* is defined as:

$$\hat{k} = \min\{k : |U_T(k)| = \max_{1 \leq j \leq T} |U_T(j)|\} \qquad (2.3)$$

The estimate $\hat{k}$ is the point at which there is maximal sample evidence for a break in the squared returns process. In the presence of a break it is proved that $\hat{k}$ is a consistent estimator of the unknown change-point k^*. It is more interesting to state the results in terms of the estimator of τ^*, $\hat{\tau} = \hat{k}/T$ with $P\{|\tau^* - \hat{\tau}| > \varepsilon\} \leq C/(\delta\varepsilon^2\sqrt{T})$, where C is some positive constant and δ depends on the ARCH parameters and $|\tau^* - \hat{\tau}| = O_p(1/T)$ (Kokoszka and Leipus, 2000). Under the null hypothesis of no break:

$$U_T(k) \to_{D[0,1]} \sigma B(k) \qquad (2.4)$$

where $B(k)$ is a Brownian bridge and $\sigma^2 = \sum_{j=-\infty}^{\infty} Cov(X_j, X_0)$. Consequently, using an estimator $\hat{\sigma}$, one can establish that under the null:

$$\sup\{|U_T(k)|\}/\hat{\sigma} \to_{D[0,1]} \sup\{B(k) : k\epsilon[0,1]\} \qquad (2.5)$$

which establishes a Kolmogorov-Smirnov type asymptotic distribution.

The computation of the Kokoszka and Leipus (2000) test is relatively straightforward, with the exception of $\hat{\sigma}$ appearing in (2.5). The authors

suggest to use a Heteroskedasticity and Autocorrelation Consistent (HAC) estimator applied to the X_j process. Andreou and Ghysels (2002) experimented with a number of estimators in addition to the procedure of Den Haan and Levin (1997) who propose a HAC estimator without any kernel estimation, which is called the Autoregression Heteroskedasticity and Autocorrelation Consistent (ARHAC) estimator. This estimator has an advantage over any estimator which involves kernel estimation in that the circular problem associated with estimating the optimal bandwidth parameter can be avoided. Related is the evidence that CUSUM tests for mean shifts when implemented using HAC estimators may exhibit non-monotonic power (Vogelsang, 1999) especially when the bandwidth parameter has been selected by data-driven methods. A new estimator for the long-run variance that is asymptotically not affected by the bandwidth parameter choice is presented in Altissimo and Corradi (2003). Although the study of the non-monotonic power for variance shifts is not considered in the present analysis it would be interesting to compare the power of volatility CUSUM tests for the den Haan and Levin (that does not involve kernel bandwidth data-driven selection) and Altissimo and Corradi long-run variance estimators.

Since the process of interest $X_t = |r_t|^\delta$ for $\delta = 1, 2$ represents an observed measure of the variability of returns we may use high frequency volatility filters e.g. $X_t = QV_t$ and PV_t, as discussed in the previous section. Recall that in the context of the SV and GARCH models $\{r_t\}$ represents a β - mixing process and that the measurable functions of mixing processes are mixing and of the same size (White, 1984, Theorem 3.49). Similarly the high frequency returns process $\{r_{(m),t}\}$ generated by (2.1) is β-mixing and the high frequency filters are $X_t = G(r_{(m),t}, .., r_{(m),t-\tau})$, for finite τ, are also β -mixing. Hence QV_t and PV_t can also represent control processes for the CUSUM test. Andreou and Ghysels (2002, 2004) present a comprehensive analysis of the historical CUSUM process for alternative control processes and change-points and show via simulations and empirical illustrations the properties of this test.

3 Asymptotic Local Power with Varying Sampling Frequencies

The usual local asymptotic analysis consists of constructing a sequence of local alternatives which converge to the null hypothesis at a rate $T^{-1/2}$. In this section we study local asymptotic power under various sampling frequency schemes. Such an analysis is to the best of our knowledge novel. Two issues are important when studying asymptotic local power and the effect of sampling frequency. First, changing sampling frequencies has an immediate impact on the sample size, and therefore also on the distance of the local alternative with respect to the null. If this would be the only effect the predictions would be rather straightforward, since more data means local alternatives closer to the null. The complicating factor is that the change in sampling frequency alters the distance between alternatives and the null due to the change in the data generating process under both the null and alternative as one changes the sampling frequency. Take for instance the case of conditional volatility models. Drost and Nijman (1993), Drost and Werker (1996) and Meddahi and Renault (2004) studied the temporal aggregation properties for weak GARCH and SV models. From their analysis we know how the volatility dynamics and the tail behavior of innovations change as the sampling frequency is changed.

Kokoszka and Leipus (2000) consider an ARCH(∞) process and their test involves using squared or absolute returns. Following their setup and to facilitate the presentation of our results we start with high frequency returns, i. e. the process of interest is $X_t = |r_{(m),t}|^\delta$ for $\delta = 1, 2$. Furthermore, suppose the parameter vector consists of $b \equiv (b_j^{(m)}, j = 0, \ldots, \infty)$ and corresponds to the high frequency sampling ARCH representation. Allowing for change-points in the parameters implies that the vector $b_{(m),t} \equiv (b_{(m),j,t}, j = 0, \ldots, \infty)$ exhibits time variation. Recall that the return process $\{r_{(m),t}\}$ follows an ARCH(∞) process, $r_{(m),t} = u_{(m),t}\sqrt{h_{(m),t}}$, $h_{(m),t} = b_{(m),0} + \sum_{j=1}^{\infty} b_{(m),j} r_{(m),t-j}^2$, $b_{(m),0} \geq 0, b_{(m),j} \geq 0, j = 1, 2$, with finite fourth moment and errors that can be non-Gaussian. The generic null can be written as follows:

$$H_0 : b_{(m),j,t} = b_{(m),j} \qquad \forall t = 1, \ldots, T \qquad \forall j \geq 1. \qquad (3.1)$$

The sequence of local alternatives assumes a single structural break at some

point in the sample, namely:

$$
b_{(m),j,t} = \begin{cases} b_{(m),j} + \beta_{(m),j}^{(1,t)}/\sqrt{mT} & t = 1,...,k^* \\ b_{(m),j} + \beta_{(m),j}^{(2,t)}/\sqrt{mT} & t = k^* + 1,...,T \end{cases}
$$

Note that the sampling frequency enters explicitly the sequence of local alternatives. With $m = 1$ we obtain the usual case of local drift. With values $m > 1$ we sample high frequency data resulting in a sample size mT and local alternatives that move with the change of frequency. We first need to list a number of regularity conditions, they are:

Assumption 3.1 *The sequence of local alternatives satisfies for each m:*

- $\beta_{(m),j}^{(i,t)} \to \beta_{(m),j}^{(i)}$ *for* $i = 1, 2$, $\forall$ j *as* $T \to \infty$.

- $\beta_{(m),j}^* := \max_{i=1,2} \sup_{T>1} |\beta_{(m),j}^{(i,t)}| < \infty$

- *Let $\kappa_{(m)}$ be the kurtosis of the ARCH(∞) process innovations, then:*

$$
\left(\kappa_{(m)}\right)^{1/4} \sum_{j=1}^{\infty} (b_{(m),j} + \beta_{(m),j}^*) < 1
$$

For the purpose of presentation we compare sampling frequency $m = 1$ with a finer sampling frequencies $m > 1$ and study the local power at sample points common to both frequencies (i.e. sample points determined by $m = 1$). Recall that in order to test for breaks in an ARCH(∞) Kokoszka and Leipus (2000) consider the following process:

$$
U_T(k) = \left(1/\sqrt{T} \sum_{j=1}^{k} X_j - k/(T\sqrt{T}) \sum_{j=1}^{T} X_j\right) \tag{3.2}
$$

where $0 < k < T$. We are interested in evaluating the local asymptotic power for various sampling frequencies m. The following result is shown in the Appendix:

Proposition 3.1 *Under the sequence of local alternatives (3) satisfying assumption 3.1:*

$$
U_{mT}(k) \to_{D[0,1]} \sigma B(k) + G_{(m)}(k) \tag{3.3}
$$

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for $k = 1, 2, \ldots, T$, $B(k)$ is a Brownian bridge and $\sigma^2 = \sum_{j=-\infty}^{\infty} Cov(X_j, X_0)$ and $G_{(m)}(k)$ is as follows:

$$G_{(m)}(k) := (k \wedge k^* - kk^*)\Delta_{(m)} \tag{3.4}$$

Moreover the relative local asymptotic power is:

$$\Delta_{(m)}/\Delta_{(1)} = \kappa_{(m)}[(\beta_{(m),0}^{(1)} - \beta_{(m),0}^{(2)})(1 - \kappa_{(m)}B_{(m)}) + \kappa_{(m)}b_{(m),0} \times$$

$$\sum_{j=1}^{\infty}(\beta_{(m),j}^{(1)} - \beta_{(m),j}^{(2)})](1 - \kappa_{(1)}B_{(1)})^2/\kappa_{(1)}[(\beta_{(1),0}^{(1)} - \beta_{(1),0}^{(2)})(1 - \kappa_{(1)}B_{(1)})$$

$$+\kappa_{(1)}b_{(1),0}\sum_{j=1}^{\infty}(\beta_{(1),j}^{(1)} - \beta_{(1),j}^{(2)})](1 - \kappa_{(m)}B_{(m)})^2 \tag{3.5}$$

where $B_{(m)} = \sum_{j=1}^{\infty} b_{(m),j}$.

It will be helpful to simplify the expression by looking at a special case of a ARCH(1) model with a break in the slope parameter and a fixed intercept across the two regimes. In particular, the following result holds:

Corollary 3.1 *Assume the ARCH(1) model*

$$h_{(m),t} = b_{(m),0} + b_{(m),1}r_{(m),t-j/m}^2$$

and no break in the intercept, then

$$\Delta_{(m)}/\Delta_{(1)} = \frac{\kappa_{(m)}^2(1 - \kappa_{(1)}b_{(m),1}^m)^2}{(1 - \kappa_{(m)}b_{((m),1)})^2\sqrt{m}b_{(m),1}^{m-1}\kappa_{(1)}^2} \tag{3.6}$$

where

$$\kappa_{(1)} = 3 + \frac{\kappa_{(m)} - 3}{m} + 6(\kappa_{(m)} - 1)\frac{[m(1 - b_{(m),1}) + b_{(m),1}^m]b_{(m),1}}{m^2(1 - b_{(m),1})^2} \tag{3.7}$$

and

$$b_{(1),j,t}^{(T)} = \begin{cases} b_{(m),j}^m + [\sqrt{m}b_{(m),1}^{m-1}]\beta_{(m),j}^{(1,t)}/\sqrt{T} + o_p(\sqrt{T}) & t = 1, \ldots, k^* \\ b_{(m),j}^m + [\sqrt{m}b_{(m),1}^{m-1}]\beta_{(m),j}^{(2,t)}/\sqrt{T} + o_p(\sqrt{T}) & t = k^* + 1, \ldots, T \end{cases}$$

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The proof appears in the Appendix. Note that the ratio in (3.6) only depends on two parameters, namely the ARCH persistence parameter $b_{(m),1}$ and the tail behavior of the ARCH process expressed by $\kappa_{(m)}$. The latter two parameters determine the relative local power trade-off.. When $\Delta_{(m)}/\Delta_{(1)}$ equals one then the test exhibits equal local asymptotic power at both sampling frequencies. In such a case the sample size reduction is exactly compensated by persistence/tail behavior trade-off.. In all other circumstances the aggregation implies a decrease in local asymptotic power, when $\Delta_{(m)}/\Delta_{(1)} > 1$ or an increase when the reverse is true.

Solutions to the system of equations (3.6) and (3.7) are presented in Table 1. The different choices of m represent the situations when sampling frequency is doubled $m = 2$ (e. g. sampling returns mid and end of day), or sampling involves weekly or monthly aggregation, $m = 5$ and 22, respectively. Two alternative approaches are adopted for solving the system. First we assume that the local asymptotic power is the same irrespective of the sampling frequency $\Delta_{(m)} = \Delta_{(1)}$ which yields solutions for $\kappa_{(1)}$ and $\kappa_{(m)}$ reported in the first two columns (for each m) in Table 1. The results show a clear trade-off between the kurtosis coefficients $\kappa_{(m)}$ and the persistence coefficient b_1 for the $m = 2, 5, 22$. It is worth noting that the relationship between $\kappa_{(m)}$ and $\kappa_{(1)}$ is not monotone. Second we assume a daily kurtosis coefficient $\kappa_{(1)}$ derived from the empirical evidence of Normality $\kappa_{(1)} = 3$ and Student's $t(7)$ (Bollerslev, 1990) $\kappa_{(1)} = 5$ (given that the unconditional t kurtosis is $\kappa = 3 + 6/(\nu - 4)$) and first solve for $\kappa_{(m)}$ in (3.7) and then for (3.6). Most solutions yield that $\Delta_{(m)}/\Delta_{(1)} > 1$ which suggest that asymptotically the local power of the test will be higher for high frequency processes. This is result appears stronger for the ratio at $m = 22$ as opposed to $m = 2$.

4   Sequential CUSUM Test for Volatility Models

Real time monitoring with CUSUM type statistics based on parameter estimates or residuals of static and dynamic linear regression models for economic time series is addressed, for instance, in Chu et al. (1996) and Leisch et al. (2000) and Zeileis et al. (2004). We consider nonlinear time series models proposed for financial asset returns, namely ARCH type processes and our analysis is not strictly parametric in that it can be applied to other strongly

dependent processes that satisfy the general conditions outlined below. Moreover, we do not consider sequential change-point tests for parameter estimates or residuals of such time series models. Instead our monitoring procedures focus on the observed process and its transformations that are driven by a dynamic volatility model, as well as high frequency volatility estimators that exhibit optimal properties for a general class of GARCH and SV processes. It is important to note at the outset that unlike the residuals of dynamic linear regression models, the residuals of GARCH models do not satisfy the Functional Central Limit Theorem (FCLT) with a Wiener process limit and therefore their boundary crossing probabilities cannot be used.

For the location-scale nonlinear time series model in (2.1) we focus on the following general conditional dynamic scale model. A random sequence $\{X_t, t \in T\}$ satisfies the following equations if there exists a sequence of i.i.d. non-negative random variables $\{\xi_t, t \in T\}$ such that:

$$X_t = \sigma(\mathbf{X}_{t-1}^0, b_j)\xi_t, \quad t \geq 1 \tag{4.1}$$

$$\sigma(\mathbf{X}_{t-1}^0, b_j) = b_0 + \sum_{j=1}^{\infty} b_j X_{t-j} \tag{4.2}$$

where $b_0 \geq 0, b_j \geq 0, j = 1, 2, ...$ The model (4.1)-(4.2) in Robinson (1991) is general enough to include the following ARCH type models: The ARCH(∞) (Engle (1982)) where

$$r_t^2 = \sigma_t^2 u_t^2, \sigma_t^2 = b_0 + \sum_{j=1}^{\infty} b_j r_{t-j}^2 \tag{4.3}$$

and the Power-ARCH(∞) (Taylor (1986))

$$|r_t|^\delta = \sigma_t^\delta |u_t|^\delta, \sigma_t^\delta = b_0 + \sum_{j=1}^{\infty} b_j |r_{t-j}|^\delta \tag{4.4}$$

where $\delta > 0$ and u_t are i.i.d. random variables with zero mean. Moreover the GARCH(p, q) model

$$r_t^2 = \sigma_t^2 u_t^2, \sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} a_j \sigma_{t-j}^2 + \sum_{j=1}^{q} d_j r_{t-j}^2 \tag{4.5}$$

can be rewritten in the form of the ARCH(∞) with positive exponentially decaying b_j under some additional constraints on the coefficients a_j, d_j (Nelson and Cao (1992)). For instance, the GARCH(1,1) model can be rewritten in the form of (4.1)-(4.2) with $\sigma(\mathbf{X}_{t-1}^0, b_j) = \sigma_t^2$, $\xi_t = u_t^2$, $X_t = r_t^2$ and the coefficients $b_0 = \alpha_0(1 + \alpha_1 + \alpha_1^2 + ...) = \alpha_0/(1 - \alpha_1), b_j = a_1^{j-1} d_1$.

The following parameter and moment condition:

$$E(\xi_0^2) < \infty \text{ and } E(\xi_0^2) \sum_{j=1}^{\infty} b_j < 1 \qquad (4.6)$$

for the general specification in (4.1)-(4.2) implies that:

(a) $\{X_t\}$ is strictly and weakly stationary.

(b) $\{X_t\}$ exhibits short memory in the sense that covariance function is absolutely summable, $\sum_{t=-\infty}^{\infty} cov(X_t, X_0) < \infty$ (Kokoszka and Leipus (2000)).

(c) $\{X_t\}$ satisfies the Functional Central Limit Theorem (FCLT). Giraitis et al. (2000) prove that as $N \to \infty$

$$S_N(t) := N^{-1/2} \sum_{j=1}^{[Nt]} (X_j - E(X_j)) \to \sigma W(t), \ 0 \le t \le 1 \qquad (4.7)$$

where $\sigma^2 = \sum_{t=-\infty}^{\infty} cov(X_t, X_0) < \infty$ and $\{W(t), 0 \le t \le 1\}$ is a standard Wiener process with zero mean and covariance $E(W(t)W(s)) = \min(t,s)$. It is interesting to note that the CLT holds without having to impose any other memory restrictions on X_t such as mixing conditions. The reason being that the condition in (4.6) implies not only (a) weak stationarity but also (b) short memory. Kokoszka and Leipus (2000) show that if, in addition, b_j decay exponentially then so does the covariance function. Finally, the FCLT for X_t holds without the Gaussianity assumption.

The ARCH type models (4.3)-(4.5) can be considered in the context of the general specification (4.1)-(4.2) for which $\xi_t = f(u_t)$ for some non-negative function f. Therefore condition (4.6) can lead to corresponding conditions for the above ARCH models. For instance, for the ARCH(∞) model $\xi_t = u_t^2$ and condition (4.6) becomes $E(u_0^4) < \infty$ and $E(u_0^4) \sum_{j=1}^{\infty} b_j < 1$.[4]

The FCLT result for X_t in (4.1)-(4.2) or equivalently $|r_t|^\delta, \delta = 1, 2$ in (4.3)-(4.5) provides the conditions for deriving the sequential CUSUM tests for dynamic scale models. In contrast to linear dynamic regression models, the residuals of GARCH models do not satisfy the FCLT. Horvath et al. (2001) show that the partial sum of the ARCH(∞) residuals are asymptotically Gaussian, yet involve a covariance structure that is a function of the conditional variance parameters. Therefore the CUSUM test can be applied sequentially to the observed strongly dependent returns and their transformations i.e. $X_t = |r_t|^\delta, \delta = 1, 2$ given that the FCLT conditions (4.6) are

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[4]Davidson (2000) shows that the FCLT holds for GARCH(p, q) processes based on the sufficient assumptions that u_t is i.i.d. with finite fourth moment in the near-epoch dependence framework.

satisfied. Consequently, boundary crossing probabilities can be computed for the CUSUM test based on X_t as opposed to the ARCH type residuals. Monitoring X_t is equivalent to sequentially testing the stability of dynamic risk measures. Such on-line tests can be considered useful for providing an early warning for a disruption in financial risk which can be considered useful for financial crises.

Let us now turn to the derivation of the sequential CUSUM test. Consider a special case of the process X_t in (4.1)-(4.2) as a simple way of specifying the sequential monitoring scheme. The returns process $r_{(m),t}$ sampled at frequency $1/m$ follows a GARCH(1,1) model:

$$r_{(m),i,t} = \sigma_{(m),i,t} \cdot u_{(m),t}$$
$$\sigma^2_{(m),i,t} = b_{0,i,(m)} + b_{1,i,(m)} r^2_{(m),t-1/m} + \gamma_{i,(m)} \sigma^2_{(m),i,t-1/m}, \quad t = 1,...n, n+1,...$$
(4.8)

where $b_{0,i,(m)} \geq 0, b_{1,i,(m)}, \gamma_{i,(m)} > 0$ and $u_{(m),t}$ is an i.i.d. zero mean process. Condition (4.6) is applied to $u_{(m),t}$ and the FCLT holds for $r^2_{(m),t}$. The subscript i allows for the possibility of a structural change in the volatility process, $\sigma^2_{(m),i,t}$. Consider the historical sample $t = 1,...,n$ where the model is homogeneous such that there are no structural changes i.e. $b_{0,i,(m)} = b_{0,(m)}$, $b_{1,i,(m)} = b_{1,(m)}$, $\gamma_{i,(m)} = \gamma_{(m)}$. The objective is to use this historical homogeneous sample to estimate the daily volatility given by $\sigma^2_{(1),i,t}$ where the subscript (1) refers to the volatility for 1 day. As mentioned above and for uniform notation we will drop the subscript "one" that refers to the sampling frequency when we refer to the daily frequency such that the daily volatility is given by $\sigma^2_{i,t}$. It is important to emphasize that we neither estimate the GARCH parameters nor use temporal aggregation of (4.8) and therefore we do not entertain in weak GARCH specifications. Instead we may deal with a strong GARCH and temporally aggregate the high frequency returns process $r_{(m),i,t}$ via intraday sums to obtain data-driven estimators for daily volatility.

High frequency volatility estimators are employed in order to filter $\sigma^2_{i,t}$ and we choose to focus on the Power and Quadratic Variations, PV_t and QV_t. Other data-driven estimators can also be used. The historical sample $t = 1,...,n$ is assumed to be homogeneous such that σ^2_t is estimated by $\{PV_t\}^n_1$ and $\{QV_t\}^n_1$. The objective is to monitor sequentially new data from time $n+1$ onwards to test whether any change-point occurs in $\sigma^2_{i,t}$ and thereby $r_{i,t}$ during the monitoring period. Hence the sequential test has the following null hypothesis $\sigma_{(m),i,t} = \sigma_{(m),t}$ for $i > n+1$ against the alternative that at some point in the future the volatility process will change to $\sigma_{(m),i,t}$ at some

point $\tau > n$.[5] Since a break in the unobserved $\sigma_{(m),i,t}$ implies a change in the observed returns process, $r_{(m),i,t}$, the following two classes of control processes are considered for sequential testing: (i) The high frequency daily volatility filters $\{PV_t\}_{n+1}$, $\{QV_t\}_{n+1}$ and (ii) the observed daily returns transformations of $\{|r_t|\}_{n+1}$ and $\{r_t^2\}_{n+1}$. It is worth emphasizing that monitoring with these processes is valid for more general dynamic conditional variance specifications. The alternative approach of estimated parameter-based sequential tests for a strict semiparametric GARCH model is pursued in Berkes et al. (2004) based on Generalized Likelihood Ratio (GLR) tests.

The sequential CUSUM test that involves the volatility control processes X_t. The sequential CUSUM test that involves the above control processes can be derived for instance for the daily squared returns process of (4.8). It follows from the FCLT that the partial sum:

$$(T - n)^{-1/2} \sigma^{-1} \sum_{j=n+1}^{\tau T} (r_j^2 - \mu) \to W(\tau) \qquad (4.9)$$

converges to the Wiener process W. Note that $\mu = E(r_j^2)$ and refers to the mean over the respective partial sum. The derivation of the sequential CUSUM test can be found, for instance, in Chu et al. (1996). Take $T = \lambda n$, then (4.9) is

$$n^{-1/2} (\lambda - 1)^{-1/2} \sigma^{-1} \sum_{j=n+1}^{\tau T} (r_j^2 - \nu) \to W(\tau). \qquad (4.10)$$

The CUSUM process for monitoring is defined by:

$$Q_T^n = n^{-1/2} \sigma^{-1} \left[(\lambda - 1)^{-1/2} \sum_{j=n+1}^{\tau T} (r_j^2 - \nu) - \sum_{j=1}^{n} (r_j^2 - \nu) \right] \to B(\tau) \qquad (4.11)$$

if $n \to \infty$ and $\lambda \to 2$ or $t/n \to 1$, where $B(\tau)$ is a Brownian bridge:

$$Q_T^n = n^{-1/2} \sigma^{-1} \left[\sum_{j=n+1}^{\tau T} (r_j^2 - \nu) - \sum_{j=1}^{n} (r_j^2 - \nu) \right] \to B(\tau) \qquad (4.12)$$

The variance $\sigma^2 = \sum_{j=-\infty}^{\infty} cov(r_j^2, r_0^2)$ and a consistent estimator for the long-run variance, e.g. a Heteroskedastic Autocorrelation Consistent (HAC) estimator for r_j^2. As mentioned in section 2.2 the ARHAC estimator is used for σ.

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[5]We assume that the error process $u_{(m),t}$ is homogeneous and does not represent another possible source of change in $\sigma_{(m),i,t}$. Such change-point alternatives are considered elsewhere, see for instance Andreou and Ghysels (2003).

For the sequential CUSUM statistic the theory of the drifted Wiener process provides a way to find the appropriate boundaries. Linear and parabolic boundary functions can be considered that grow with the sample size at approximately rates $\sqrt{t}$ and $\sqrt{\log t}$.

Certain boundary functions for the Brownian bridge are derived using the Law of Iterated Logarithm (LIL). For instance, Khintchine's LIL shows that the boundary $b_1(t) = \sqrt{2t \log \log t}$ standardizes the supremum of the Brownian motion. This LIL extends to partial sums of dependent processes (Altissimo and Corradi, 2003, Segen and Sanderson, 1980) and is motivated by the fastest detection of change. However, Altissimo and Corradi (2003) and Chu et al. (1996) show via simulations that in finite samples the probability of type one error is erratic and high for $b_1(t)$. Chu et al. (1996) also extend the i.i.d. results of Robbins and Seigmund (1970) to monitoring procedures where the partial sum of the residuals and recursive estimates in linear time series obey the FCLT, under milder restrictions on general class of boundary functions (see Theorem 3.4, p. 1051 and Theorem A, Appendix A, p. 1062). The following boundary is derived analytically for the sequential CUSUM-type and Fluctuation tests:

$$b_2(t) = \sqrt{t(t-1)\left[\alpha^2 + \ln\left(\frac{t}{t-1}\right)\right]} \qquad (4.13)$$

where $\alpha^2 = 7.78$ and 6.25 gives the 5% and 10% monitoring boundaries, respectively.

Revesz (1982) shows that the asymptotic growth rate of the supremum of the increments of the Wiener process is $\sqrt{\ln t}$ almost surely. Leisch et al. (2000) specify the boundary $b_3(t) = \lambda\sqrt{2\log_+ t}$ where $\log_+ t = 1$ if $t \le e$ and $\log_+ t = \log t$ if $t > e$, for which the critical values λ for the increments of a Brownian bridge are obtained via simulations in the context of Generalized Fluctuation tests (see for instance, Table 1, p. 846, Leisch et al., 2000). Similarly, Altissimo and Corradi (2003) suggest $\lambda\sqrt{2\ln(\ln t)}$ based on the exact distribution of the supremum of a Brownian bridge, where λ is derived via simulation to yield the adjusted bounds (reported in Table 2, Altissimo and Corradi, 2003). Leisch et al. (2000) show that $b_3(t)$ leads to tests that are sensitive to change-points that occur early or late in the monitoring period as opposed to $b_2(t)$. Zeileis et al. (2004) argue that simulation evidence for $\lambda\sqrt{\log_+ t}$ suggests most of the size of the test is used at the point where the boundary changes from being constant to growing and this makes it inap-

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propriate for a process with growing variance such as the Brownian bridge. Hence they suggest a simple boundary that has the advantage of not using up the size of the corresponding test at the beginning of the monitoring period while at the same time growing in a linear manner given by

$$b_4(t) = \lambda t \tag{4.14}$$

where λ is the simulated critical value for alternative monitoring horizons (shown in Table III in Zeileis et al., 2004). In the remainder of the analysis we consider $b_2(t)$ and $b_4(t)$ and note that these linear boundaries cross at some point during the monitoring period. For instance, if the historical sample is n then $b_2(t)$ and $b_4(t)$ cross at $2n$ for a 10% significance level. After the crossing point the slope of $b_4(t)$ is lower than that of $b_2(t)$ which implies that it is more likely to capture small breaks late in the sample. In contrast, before the crossing point $b_4(t)$ rests higher then $b_2(t)$ which means that the probability to detect an early break is lower for $b_4(t)$ relative to $b_2(t)$.

The choice of the boundary in sequential testing is traditionally followed based on the minimization of the average detection delay of the Average Run Length (ARL). Other criteria or the combination of boundaries or the derivation of optimal tests given a boundary are still open questions in the literature.

5 Simulation evidence for sequential CUSUM

In this section we examine the properties of the sequential CUSUM change-point test for GARCH processes. The analysis has the following objectives: (i) The finite sample properties of the monitoring procedure are examined for alternative boundaries and volatility filters and daily returns processes. (ii) The effects of heavy tails and persistence in a GARCH on the CUSUM test are examined in conjunction with the local power results in a sequential framework. The results in this section extend existing simulation evidence in the following respects: First, results are presented for the finite sample behavior of the on-line CUSUM test for strongly dependent processes as opposed to the independent models presented in Chu et al. (1996) and Leisch et al. (2000) and linear time series models in Zeileis et al. (2004). Second, supportive evidence is provided for the CUSUM test for monitoring directly the

observed returns process and its transformations as well as high frequency volatility filters instead of adopting a more traditional approach of monitoring the parameter estimates of a volatility model or its residuals. Third, it introduces another important variable in sequential testing namely the sampling frequency and shows how high frequency volatility filters can improve the detection delay of the CUSUM test for GARCH models to rates that are comparable to NIID setups (of Chu et al., 1996 and Leisch et al., 2000).

5.1   The Monte Carlo design

The simulated returns process $r_{(m),t}$ sampled at frequency $1/m$ is generated by a GARCH(1,1) model:

$$r_{(m),t} = \mu_{(m),t} + \sigma_{(m),t} \cdot u_{(m),t}$$
$$\sigma^2_{(m),t} = b_{(m),0} + b_{(m),1} r^2_{(m),t-1/m} + \gamma_{(m)} \sigma^2_{(m),t-1/m}, \quad t = 1, ..., T. \tag{5.1}$$

where $u_{(m),t}$ is $i.i.d.$ $N(0,1)$ and $\sigma^2_{(m),t}$ is the volatility process. The Data Generating Processes (DGP) at the 30-minute frequency are: (i) a low persistent GARCH where $b_{(m),0} = 0.038$, $b_{(m),1} = 0.056$, $\gamma_{(m)} = 0.933$ and (ii) a high persistent GARCH $b_{(m),0} = 0.034$, $b_{(m),1} = 0.035$, $\gamma_{(m)} = 0.96$, both of which can be considered representative processes for financial asset returns. Note that at the 30-minute frequency $m = 48$ for the 24-hour traded markets. Using the 30-minutes returns process in (5.1) we temporally aggregate them to the daily frequency without imposing the assumption that at those frequencies the data are driven by a GARCH process with $i.i.d.$ errors. The reason being that we do not wish to impose the strong GARCH process assumption (Drost and Nijman, 1993) at all frequencies since these do not temporally aggregate.

The control processes refer to two classes: (i) the observed daily returns process and its square and absolute transformations, r_t^2 and $|r_t|$ and (ii) high frequency volatility estimators, namely QV_t and PV_t. Barndorff-Nielsen and Shephard (2004) show that the asymptotic properties of $QV_t^{(m)}$ and $PV_t^{(m)}$ are valid even if $m = 48$.

Under the null, the process in (5.1) is driven by a homogeneous process without breaks in the conditional variance.[6] We compute the empirical crossing probabilities under the null for historical sample sizes $n = 75, 125, 250, 500$

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[6]It is assumed that $\mu_{(m),t} = 0$ for simplicity. Consequently, we focus on scale change point alternatives that are more challenging to detect as opposed to mean shifts.

days. The choice of the historical sample takes into account various financial applications e.g. one or two years of daily data are recommended by regulators for estimating risk management measures. Smaller samples can also be considered when there is an end or beginning of sample instability. Such small historical samples are feasible for the control processes considered in the analysis (i.e. daily returns transformations and high frequency volatility filters) but for the estimation of GARCH type models and for monitoring its parameters larger samples are preferred. The monitoring horizon q is set to $q = 2n, 4n, 6n, 8n, 10n$ days. According to the theory if n is large enough and q is extended to infinity the crossing probabilities should approach the nominal size for the given boundaries $b_2(t)$ and $b_4(t)$. Although it is not possible to set q equal to infinity in simulations, the objective is to examine whether there are any size distortions for the above test with various finite values of q.

Under the alternative hypothesis the returns process is assumed to exhibit a change-point in the conditional variance, $\sigma_{(m),t}$, which can be thought as permanent regime shifts in volatility at an out-of- sample point $\tau = \pi n$ (where $\pi = 1.1$ and 3 represent respectively beginning- and end-of-sample instabilities) and $q = 10n$. Such breaks may be due to an increase in the intercept, $b_{(m),0}$, or a shift in the volatility persistence, $b_{(m),1} + \gamma_{(m)}$ or both. For both low and high persistent GARCH processes under the change-point regime the constant shifts from $b_{(m),0}$ to $3b_{(m),0}$ and the dynamics from $b_{(m),1} = 0.94$ to $b_{(m),1} = 0.91$ in the high persistence GARCH and $b_{(m),1} = 0.933$ to $b_{(m),1} = 0.91$ in the low persistence GARCH. Under the alternative hypothesis it is useful to examine whether the test is consistent i.e. it eventually signals the break, yet it is even more informative to show that the on-line CUSUM, given a boundary, minimizes the detection delay which is defined at the Average Run Length (ARL). Similarly, the empirical distribution of the first hitting time gives a more complete picture of the power properties of the test. All simulation results are based on 1000 trials.

5.2 Simulation results

The simulation results relating to the size of the sequential CUSUM in (4.12) can be found in Tables 2 and 3 for the two classes of control processes, the daily returns transformations, r_t^2 and $|r_t|$ and the high frequency volatility estimators, QV_t and PV_t. The size properties are examined for both low and high persistent GARCH models and for various historical samples, n,

and monitoring horizons, q, as defined above. We examine the two linear boundaries: The Chu et al. $b_2(t)$ boundary in equation (4.13) and the Zeileis et al. boundary $b_4(t)$ in equation (4.14), at the 10% significance level.

Table 2 shows that for both low and high persistence GARCH processes and for the two boundaries considered and for the historical sample, n, the size of the on-line test for r_t^2 and $|r_t|$ behaves as follows:

- Across *boundaries* the size of the test improves with large $n = 250, 500$ and $q > 6n$. Size distortions are higher for $b_4(t)$ than for $b_2(t)$. The Chu et al. boundary has empirical size close to the nominal even for small $n \geq 125$ and $q > 2n$ for $|r_t|$ in the context of a low persistence GARCH. For high-persistence GARCH models the CUSUM test yields size distortions almost twice the nominal size and is not recommended for small historical samples of $n = 75$.

- Across *control processes* the on-line CUSUM has relatively more size distortions for the r_t^2 than for the $|r_t|$ process (with its size being around 5% larger for $n = 250, 500$).

- For the low persistence GARCH process and $b_2(t)$ the $|r_t|$ yield empirical size very close to the nominal for different choices of n and q.

Table 3 also presents the size of the sequential CUSUM test for the second class of monitoring processes, the high-frequency volatility filters, QV_t and PV_t, that behaves as follows:

- For the PV_t the size of the sequential CUSUM improves for $n \geq 250$ and $q \geq 4n$ and either boundary. For $n = 500$ and $b_2(t)$ the size is close to the nominal one.

- Across control processes the test has relatively more size distortions for the QV_t as opposed to the PV_t process.

Comparing the simulated size of the sequential CUSUM for the two classes of monitoring processes we find that the test has less size distortions for the high-frequency volatility filters as opposed to the daily returns transformations. This can be explained by the fact that high-frequency volatility filters are more accurate and smooth estimates of volatility as opposed to the absolute and squared returns. For alternative boundaries, GARCH persistence

effects, n and q, the monitoring process with size close to the nominal one is $|r_t|$ followed by PV_t. For the same parameters the size of r_t^2 and QV_t is on average 1.5 times larger than the nominal one.

We now turn to assess the power of the sequential CUSUM test by evaluating the empirical distribution of the first hitting time. These simulation results are also compared across different boundaries, change-point locations and control processes as follows:

Properties of ARLs

The detection timing or delay is measured by the Average Run Length (ARL). The size and type of break affects the ARL in the following way. Although in the simulation setup both breaks in $b_{0,(m)}$ and $b_{1,(m)}$ lead to the same magnitude of change of the unconditional variance $(b_{0,(m)}/(1 - b_{1,(m)}))$, the size of change in the constant is of course larger than that of the dynamics of the GARCH. More importantly the break in the constant leads to a level shift whereas a break in the dynamics leads to a trend or slope change which may be relatively more difficult to detect. Hence the ARL is lower when there is a break in the constant as opposed to the dynamics of a GARCH. The simulation results also provide evidence that this result is valid for both linear boundaries, classes of control processes, locations of break and GARCH level of persistence. More specifically we find that when there is a shift in the constant the ARL is half the size of the ARL obtained when there is break in the dynamics of a GARCH. Within the same setup (of control processes, linear boundaries and change-points) the ARLs are small and comparable to the ARLs for the NIID case reported in Chu et al. (Table II). For instance, Table 4 shows that monitoring the PV_t and QV_t with the Chu et al. boundary when there is a break at $\tau = 1.1n$ in the constant of the GARCH leads to an *average* ARL of 24 days across n. In the same setup when there is a break in the dynamics of the GARCH the choice of monitoring the PV_t yields a lower *average* ARL of 44 days as opposed to 90 days when monitoring the QV_t.

Dispersion of hitting times

The standard deviation of the first hitting time decreases significantly for $n \geq 75$ for an early change point at $\tau = 1.1n$ in the constant of a low persistence GARCH for the QV_t and PV_t and for both linear boundaries, as shown in Tables 4 and 7, respectively. The same result holds for the r_t^2 and $|r_t|$

when $n \geq 250$ as shown in Tables 8 and 9. A lower standard deviation implies a lower probability of false alarm. The larger historical sample $n \geq 250$ is required to reduce the standard deviation when the underlying GARCH exhibits higher persistence, as shown in Table 5. However, when there is a break in the dynamics of the GARCH the standard deviation of the first hitting time increases dramatically (as opposed to when there is break in the constant) for all processes except for the PV_t and $|r_t|$ when $\tau = 1.1n$. Their standard deviation appears stable when $n = 500$ for an early change-point and for alternative boundaries, persistence effects and types of change-point. This result suggests that the precision of the on-line CUSUM depends on how accurately we estimate the conditional variance in a historical sample.

The efficiency of high-frequency volatility filters PV_t and QV_t is due to the fact that they exploit the intraday information of returns and have a lower measurement error as opposed to the daily returns processes $|r_t|$ and r_t^2. On the other hand, the standard deviation appears to be high and does not improve with n when $\tau = 3n$ for both control processes and linear boundaries and GARCH models.

Choice of volatility filter

The above points also imply that one important factor for the power properties of the sequential CUSUM is the choice of monitoring or control process. The historical CUSUM test in Kokoskza and Leipus (2000) is derived for the daily returns transformations r_t^2 and $|r_t|$ from the GARCH model. The simulation results for the sequential CUSUM test for the daily returns processes are shown in Tables 8 and 9 for the Chu et al. and Zeileis et al. boundaries, respectively. The overall results suggest that across n the *average* ALR is very similar for r_t^2 and $|r_t|$ for the Chu et al. boundary equal to 50 and 53 days, respectively, when there is a break in the constant at $\tau = 1.1n$.

For the same setup if we were to consider the industry benchmark for volatility i.e. the *RiskMetrics* (RM_t) in Table 6 we find that the *average* ARL increases to 70 days. Even for $n = 500$ the RM_t yields a detection delay of 59 days as opposed to 50 and 43 days for $|r_t|$ and r_t^2, respectively, when there is a break in the constant at $\tau = 1.1n$.

The monitoring processes of high frequency volatility filters lead to the relatively lowest ARLs. For the PV_t and QV_t the *average* ARL is 24 days as shown in Table 4. Similarly they yield the lowest standard deviation of the empirical distribution of the first hitting time among all the control processes

considered. The *average* standard deviation for $b_2(t)$ and a constant change-point at $\tau = 1.1n$ is 23, 32, 74, 115, 139 days the for PV_t, QV_t, $|r_t|$, r_t^2, RM_t, respectively.

<div align="center">*Choice of Boundaries*</div>

One of the most important factors of optimality in sequential testing is the choice of boundaries. In this context we consider the traditional measure of optimality which is the earliest detection timing among two boundaries, $b_2(t)$ and $b_4(t)$. For the early change point at $\tau = 1.1n$ and all n the *average* ALR is very similar for r_t^2 and $|r_t|$ for the two linear boundaries: The ARL is equal to 50 and 53 days, respectively, for $b_2(t)$ and equal to 56 and 51 days, respectively, for $b_4(t)$ when there is a break in the constant. For the same change-point the *average* ARL for PV_t and QV_t is 24 days for $b_2(t)$ and 32 and 27 days, respectively, for $b_4(t)$. These boundaries are also evaluated with respect to the *location* of the break. The reported simulation results in Chu et al. their sequential test is less effective when $\tau = 1.2n$. This is further evaluated by Leisch et al. for $\tau = 3n$ who propose the log boundary $b_3(t)$ that has better properties for end-of-sample instability. Our simulation analysis verifies that when $\tau = 3n$ the Chu et al. boundary yields a much higher ARL than the Zeileis et al. boundary which has a lower slope as the monitoring sample increases. For instance, when there is a break in the constant at $\tau = 3n$ the *average* ARL across n is for the PV_t and QV_t are 103 and 77 days, respectively, for $b_2(t)$ and 64 and 52 days, respectively, for $b_4(t)$. Similar results extend to the case of monitoring $|r_t|$ and r_t^2 for which the *average* ARL is at $\tau = 3n$ equal to 202 and 156 days, respectively, for $b_2(t)$ and 129 and 119, respectively, for $b_4(t)$. Hence, the Zeileis et al. boundary yields the lowest ARL when used to monitor PV_t and QV_t for breaks that occur late in the sample. The symmetry of the empirical distribution of the stopping time is also evaluated for the alternative boundaries. For an early change-point the Chu et al. boundary has a symmetric distribution for the PV_t and QV_t for $n \geq 125$ and for the r_t^2 and $|r_t|$ for $n \geq 250$. The simulated distribution of the first hitting time appears to be relatively less symmetric for the Zeileis et al. boundary.

5.3   The effects of sampling frequency

The local power analysis for the historical CUSUM test in section 3 shows that the sampling frequency yields some interesting power considerations for

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the change-point analysis. Although it is less obvious how to extend these analytical results in a sequential framework given the existence of a non-constant boundary, we resort to the following simulation analysis in order to obtain some indication of the effects of sampling frequency on the on-line CUSUM. Assuming that locally the boundary is approximately constant we expect from the local power results in section 3 that if we double the sampling frequency of squared returns i.e. sample and thereby monitor twice as often then the power of the test will improve under the assumption of a low persistence ARCH with Normal tails. On the other hand, if the low persistence ARCH exhibits heavy tails then there is a trade-off between sampling frequency and power such that in the presence of a heavy tailed error GARCH the power deteriorates with sampling and monitoring twice more frequently. In order to examine these effects we generate the low persistence GARCH process with Normal and Student's $t(6)$ errors and examine the ARL for daily and half-daily control processes. If the historical local power results transfer to the sequential framework then we would expect that the ARL would improve when monitoring *half-daily* squared returns of a Normal GARCH and when monitoring *daily* squared returns of a Student's t GARCH. Note that the historical local power analysis in section 3 is derived for the squared returns process but may also extend the simulation analysis to the absolute returns transformation.

The simulation results in Tables 10 and 11 present significant evidence that support the historical local power analytical results in the sequential framework. Table 10 presents the simulated size of the absolute and squared returns processes for half-daily monitoring when the 30-minute GARCH process is driven either by Normal or $t(6)$ errors. Compared with the daily monitoring empirical size results in Table 2 we conclude that there are no size distortions from doubling the frequency. Table 11 panel (A) presents the ARLs *in days* for daily and half-daily monitoring when there is a change point in the constant of a low persistence GARCH process. There is supportive simulation evidence for the following propositions: (i) In the context of a low persistence Normal GARCH sampling and monitoring half-daily improves the ARLs of the CUSUM test for both control processes, r_t^2 and $|r_t|$ and both types of linear boundaries, $b_2(t)$ and $b_4(t)$. Under the Normal GARCH process, the *average* ARL for the daily monitoring is 51 and 52 days for the daily r_t^2, for the Chu et al. and Zeileis et al. boundaries. These ARLs improve when the sampling frequency doubles: The *average* ARL for the half-daily monitoring is 35 and 40 days for the half-day r_t^2, for $b_2(t)$ and

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$b_4(t)$, respectively. Similar results are obtained for the $|r_t|$ process. However under the Student's t GARCH process, we obtain the opposite picture, namely the *average* ARL is 52 and 106 days for daily and half-daily r_t^2 monitoring, respectively, under $b_2(t)$. Similar results obtain for $|r_t|$ and the other boundary. Moreover, these results extend to the high-frequency volatility filters for which we find that there is a strong trade-off between heavy tails and sampling frequency: For a $t(6)$ GARCH process the CUSUM test with either linear boundary, the daily ARLs for both daily PV_t and QV_t are approximately half the size of the ARLs for the half-daily high-frequency volatility control processes. Refer, for instance, to the second panel in Table 11.

6 Prediction-based formulas for earlier detection

The purpose of this section is to present a prediction-based formula that enhances the detection of change points. One of the main challenges of change-point analysis is the reduction of the delay time in detecting breaks. In the previous section we showed that in a sequential framework the CUSUM test signals the instability of the constant in a GARCH process with an ARL that is comparable to the ARLs of independent processes. However, the detection of a small break in the dynamics of a GARCH is more difficult partly due to the nonlinear time series structure of the process. The challenge remains in finding a method or a test that minimizes the detection delay and can be used in the context of prediction analysis. Here we present what is to the best to our knowledge a novel approach in reducing the ARL via a new formula which provides an estimate of the probability that a CUSUM statistic will cross a given boundary over a particular horizon. The formula yielding the probability forecast is based on the extrapolation of the current level of the CUSUM statistic over a particular horizon and exploits the local behavior of a Brownian bridge process. We find that for certain processes this sequential prediction-based method signals the change point earlier than the traditional sequential monitoring statistic. Moreover it also seems to avoid false alarms as under the null the statistic behaves well. We present the formula in a first subsection, followed by simulation evidence reported in a second subsection.

6.1 Prediction-based formula

This section presents a simple simulation design for the sequential prediction-based of say qs days ahead which yields the probability that a statistic will exceed a given boundary. The statistic relies on a scaled version of the CUSUM vis-á-vis a future boundary threshold. The approach is different from conventional sequential analysis which relies on the fact that CUSUM statistics have asymptotic distributions that are described by Brownian bridge processes. It is the properties of the Brownian bridge that determine the boundaries in sequential analysis, as discussed in detail in the previous sections. We could view this conventional approach as a global one, in contrast to a local analysis which is pursued here.

A Brownian bridge process is determined by the end-points of the path. In conventional sequential analysis the end-point is computed under the null using a pre-monitoring sample consistent estimator. Here, we do not use a long run consistent end-point, but rather a local extrapolation of the process as the end-point of a Brownian bridge process that should approximate the immediate horizon of the CUSUM statistic. The immediate horizon is typically 1, 5 or 10 days, namely the period relevant for most risk management purposes. The idea of the local approximation is to take the current level of the CUSUM statistic and extrapolate its behavior over the short horizon. In particular, suppose that the CUSUM statistic is $Q_{q_0}^n$ at a given time point q_0 greater than the historical sample $n < q_0 < \tau$. Next, consider a prediction horizon defined by q_1, which as noted earlier typically is for instance 5- and 10-days. Then consider:

$$\widehat{Q}_{q_1}^n = \left[\left(\frac{q_1 - n}{q_0 - n} \right) \sum_{j=n+1}^{q_0} x_j - \sum_{j=1}^{n} x_j \right] \tag{6.2}$$

which represents the extrapolation of where Q_t^n will be at horizon q_1. This prediction is based on the scaling of $Q_{q_0}^n$ by a factor $(q_1 - n)/(q_0 - n)$. The re-scaled CUSUM statistic appearing in (6.2) is viewed as the end-point of a Brownian bridge process which describes the behavior of the original CUSUM statistic between q_0 and q_1. Given this end-point we take the boundary for the CUSUM statistic, where the boundary is taken from the conventional sequential analysis using the (global) Brownian bridge asymptotics. Given this boundary, we can compute the probability that the CUSUM statistic tied down between $Q_{q_0}^n$ and $\widehat{Q}_{q_1}^n$ will cross the boundary level at the end-point q_1, denoted B_{q_1}. This probability can readily be computed (see e.g.

Karatzas and Shreve (1991) or Revuz and Yor (1994)):

$$P(\sup_{q_0 < t < \tau} |Q_t^n| > B_{q_1}) = \exp\left[-\frac{2[B_{q_1} - \widehat{Q}_{q_1}^n][B_{q_1} - Q_{q_0}^n]}{q_1 - q_0}\right] \qquad (6.3)$$

which represents the probability that the CUSUM statistic will exceed the boundary at q_1, i.e. the prediction horizon, given that it is a tied down Brownian motion between its current level and its extrapolated level (6.2). Note that the statistics $\widehat{Q}_{q_1}^n$ and $Q_{q_0}^n$ are standardized by the HAC estimator that involves the respective q_i sample.

It is worth noting that the formula appearing in (6.3) provides certain advantages when compared to the traditional sequential analysis. In particular, it yields a 'probability statement' about the likelihood of change-point occurrences. As long as a CUSUM statistic has not crossed a boundary there is no obvious mechanism to make a statement about the likelihood of a break. The formula presented in this section remedies this. It is clear, however, that more work is required on this topic and the purpose of the next subsection is to show that the suggestion to inflate the CUSUM statistic as in (6.2) appears to work. One drawback, however, is that the horizon chosen may be perceived as ad hoc and so does the choice of inflation method for the CUSUM statistic. In addition, we do not have a clear idea yet of the asymptotic distribution of the crossing probabilities under various alternatives. Bootstrapping the probability appearing in (6.3) is one possibility. We leave this subject for future research.

6.2 Simulation results

We examine the properties of the above sequential predictive statistic using a small simulation design. As in the previous section we first examine the sequential CUSUM test and the ARL for the low persistence GARCH process in 5.1). The results in Table 12 show the ARLs for the two alternative change-point sources (a break in the constant and the dynamics of a GARCH), the linear boundaries in Chu et al., $b_2(t)$, and Zeileis et al., $b_4(t)$, and the two classes of control processes based on daily returns transformations and high frequency volatility filters. The historical sample involves around one year of daily observations, $n = 250$, for a total sample of $q = 10n$ observations and the break occurs at $\tau = 1.5n$. It is evident from the results in Table 12 that in all cases the ARLs of the sequential CUSUM statistic as analyzed

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in section 5, are relatively high especially for break in the dynamics of a GARCH. Therefore we investigate whether the simple sequential prediction-based formulae in (6.2)-(6.3) result in probability predictions that yield early warning signals of disruptions such that the detection delay is less than that of sequential tests. In the context of the same simulation setup as in Table 12 we set the following parameters: The starting date of the prediction-based analysis is set before the change-point, $q_0 = 1.3n$, the prediction horizons are 5- and 10-days ahead $q_1 = q_0 + p_f$ where $p_f = 5$ or 10 days and the prediction sample is equal to 100 observations. Alternative choices of these parameters are considered in order to robustify the results discussed below. The sequential prediction-based test is simulated under the null and under the two alternative hypotheses of a change-point in the constant and dynamics of a GARCH process. The time series plots of the forecast probabilities are presented. Under the change-point hypothesis the sequential prediction probabilities start trending upwards shortly after the break.

For all the above choice parameters of the sequential prediction-based design we find that the daily $|r_t|$ instead of r_t^2 yields the most robust and relatively better prediction-based results as compared with high frequency volatility filters and squared returns. Figures 1 and 2 present the simulated 100 sequential probabilities of $|r_t|$ for the 10-days ahead forecasts of the Chu et al. and Zeileis et al. boundaries, respectively. Under the null the sequential probabilities for $|r_t|$ fluctuate around a constant mean of 0.007 and 0.005 for $b_2(t)$ and $b_4(t)$, respectively. The maximum simulated probability under the null hypothesis defined by $\max p_0$ is presented in Table 13. The maximum of the forecast probabilities under the null for: (i) r_t^2 does not exceed 0.015 and 0.016 and for (ii) $|r_t|$ does not exceed 0.013 and 0.01, for $b_2(t)$ and $b_4(t)$, respectively. Under the alternative hypotheses of a change-point in the constant or the dynamics of a low persistent GARCH process the forecast probabilities, p_1, present dominant trends that deviate from the probabilities under the null as shown in Figures 1 and 2. The interesting feature is that the change in the trend of these forecasts occurs close of the simulated change-point which is observation $\tau = 50$ on the graphs. Although bootstrap confidence intervals are expected to give sharper inferences we may also consider the graphical analysis of these forecasts in relation to their summary in Table 13. For $|r_t|$ we find that the forecast probabilities for a change in the constant of a GARCH begin to trend around observation 60 for both types of boundaries and forecast horizons. The top panel of Table 13 shows the exact predicted time, τ_p, for which the forecast probabilities of absolute returns,

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p_1, exceed the maximum of the simulated sequential probabilities under the null, $\max p_0$. It is worth noting that the sequential prediction-based analysis for $|r_t|$ when there is a break in the constant warns that the probabilities will exceed the given boundary with an *average* detection delay of 9 to 12 days for $b_2(t)$ and $b_4(t)$, respectively. Compared with the sequential tests we find that for $|r_t|$ the prediction-based formula improves the ARL by an average of 70 to 64 days for $b_2(t)$ and $b_4(t)$, respectively. Similar results are valid for squared returns when there is a change point in the constant of a GARCH. When there is a break in the dynamics of a GARCH the predicted probabilities for absolute returns exceed the maximum simulated probability under the null at points 78 and 80 for $b_2(t)$ and $b_4(t)$, respectively, shown in Figures 1 and 2. It is striking that the prediction-based method yields a detection delay that is around half of the ARL in sequential testing, as shown in Table 12, for daily returns processes. However, the prediction-based analysis for a change in the dynamics of a GARCH are less robust and we find that the absolute daily returns yield relatively better results than the squared daily returns.

The prediction-based analysis using high frequency volatility filters not only provides significant improvement for the detection delay when there is a change-point in the constant of a GARCH but also when there is a break in the dynamics. Moreover, robust prediction-based probabilities are provided for both PV_t and QV_t. Figures 3 and 4 present the simulated 100 forecasts for PV_t and QV_t, respectively, for a 10-day ahead forecasting horizon and the Zeileis et al. boundary. The sequential prediction-based probabilities under the change-point alternatives show a dominant trend that differs significantly from that under the null hypothesis. Moreover, under the null the simulated probabilities for PV_t and QV_t have the same range and similar $\max p_0$ as that of $|r_t|$ and r_t^2. The detection delay is significantly improved with sequential prediction-based CUSUM. When there is a change-point in the constant the *average* ARL is 8 days for the PV_t and QV_t for $b_2(t)$ and $b_4(t)$. This improves the ALR of the sequential CUSUM test by 35 and 29 days for the PV_t and QV_t, respectively, and either boundary. Impressive reduction of the detection delay occurs when forecasting a change-point in the dynamics of a GARCH. For the Zeileis et al. control limit the *average* ARLs from the prediction-based analysis are 25 and 37 days for the PV_t and QV_t, respectively, and that improves the sequential test ARLs by 52 and 181 days, respectively. Last but not least, we note that the prediction-based results under the alternative yield a change in the trend of the probabilities. Hence another possibility

to establish the structural change using the prediction-based CUSUM is to test the null of a break in the trend of probabilities. We leave this topic for future research.

7 Conclusions

Most statistical and econometric methods of sequential change-point analysis have focused on linear regression models or linear dynamic models with weak dependence. This paper addresses historical and sequential change-point tests for strongly dependent processes and in particular processes describing volatility dynamics. The interest in monitoring on-line such processes has been emphasized in recent work on quality control for risk management (Andreou and Ghysels (2003)). The present analysis focuses exclusively on CUSUM-type tests and explores the effect of sampling frequency on the local power of CUSUM tests. Analytical asymptotic local power results are derived for historical CUSUM tests showing the non-trivial trade-off between local power, sampling frequency, persistence and tail behavior. Simulation evidence supports these results for sequential tests. In addition we show that data-driven and high frequency monitoring processes for volatility have several the advantages in sequential change-point analysis. In particular, the realized power variation appears to perform relatively better among the class of high frequency data-based processes for monitoring breaks in volatility. This is not surprising given their robustness with regards to jumps, as emphasized in Barndorff-Nielsen and Shephard (2003), Ghysels et al. (2003) and Woerner (2004). The choice of boundary depends of course on the location and size of the break. We find that the power of the CUSUM test is very similar even for small change-points in the conditional variance for the linear boundaries proposed by Chu et al. (1996) and Zeileis et al. (2004). Regarding the location of the change-points, the simulation results hint towards an optimal combination of these two linear boundaries such that power is improved for both early and late breaks in the sample. The third and final contribution pertains to a new prediction-based formula that uses a local Brownian bridge approximation argument and is shown to (1) substantially reduce the delay in detection time of structural breaks and (2) provide a probability statement about the likelihood of the occurrence of a structural break.

A Proof of local asymptotic power results

The proof of Proposition 3.1 follows directly from Kokoszka and Leipus (2000) who show that under the sequence of local alternatives that satisfy the regularity conditions 3.1:

$$\Delta_{(a)} = \frac{\kappa_{(a)}[(\beta_{(a),0}^{(1)} - \beta_{(a),0}^{(2)})(1 - \kappa_{(a)}B_{(a)}) + \kappa_{(a)}b_{(a),0}\sum_{j=1}^{\infty}(\beta_{(a),j}^{(1)} - \beta_{(a),j}^{(2)})]}{(1 - \kappa_{(a)}B_{(a)})^2}$$

(A.1)

who show the result holds for $a = 1$. It is relatively straightforward to also show this holds for $a = m > 1$ which yields the desired result when the ratio of $\Delta_{(m)}/\Delta_{(1)}$ is computed. In the remainder of this appendix we focus on showing the results in Corollary 3.1. First, in the case of an ARCH(1) process without break in the intercept the ratio of relative local power simplifies to:

$$\Delta_{(m)}/\Delta_{(1)} = \frac{\kappa_{(m)}^2[b_{(m),0}(\beta_{(m),1}^{(1)} - \beta_{(m),1}^{(2)})](1 - \kappa_{(1)}b_{1(1)})^2}{\kappa_{(1)}^2[b_{(1),0}(\beta_{1(1)}^{(1)} - \beta_{1(1)}^{(2)})](1 - \kappa_{(m)}b_{(m),1})^2}$$

(A.2)

Next, we need to establish the relation between local drifts for the two sampling frequencies. The aggregation results for ARCH(1) processes are relatively straightforward (see e. g. Drost and Nijman, 1993) implying that:

$$b_{1(1),t}^{(T)} = \begin{cases} (b_{(m),1} + \beta_{(m),1}^{(1,t)}/\sqrt{mT})^m & t = 1, ..., k^* \\ (b_{(m),1} + \beta_{(m),1}^{(2,t)}/\sqrt{mT})^m & t = k^* + 1, ..., T \end{cases}$$

The local alternatives can be rewritten under both regimes as:

$$b_{(m),1}^m + [\sqrt{m}b_{(m),1}^{m-1}]\beta_{(m),1}^{(i,t)}/\sqrt{T} + o_p(\sqrt{T}) \quad i = 1, 2$$

(A.3)

yielding the result in (3.1). The result in (3.6) follows after some algebraic computations. Finally, to show that

$$\kappa_{(1)} = 3 + \frac{\kappa_{(m)} - 3}{m} + 6(\kappa_{(m)} - 1)\frac{[m(1 - b_{(m),1}) + b_{(m),1}^m]b_{(m),1}}{m^2(1 - b_{(m),1})^2}$$

(A.4)

we use the results in Drost and Werker (1996) who compute the kurtosis of both the ARCH innovations and the kurtosis in the return process. In the ARCH(1) case, however, both coincide and are equal to the above expression.

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Table 1: Numerical Solutions for the Local Power of the historical CUSUM
test for an ARCH(1) process

m	b_1	$\kappa_{(m)}$ for $\Delta_{(m)} = \Delta_{(1)}$	$\kappa_{(1)}$	$\Delta_{(m)}/\Delta_{(1)}$ for $\kappa_{(1)} = 3$	$\kappa_{(1)} = 5$
$m = 2$	0.1	32.0	27.891	5.863	18.405
	0.2	8.683	11.748	2.357	9.576
	0.3	4.126	7.841	1.130	4.064
	0.4	2.514	6.187	0.475	0.445
	0.5	3.288	11.724	0.103	0.476
	0.6	1.343	4.409	0.011	2.972
	0.7	1.109	3.449	0.470	6.423
	0.8	1.009	2.289	2.896	13.273
	0.9	0.995	1.444	20.753	47.383
$m = 5$	0.1	1996.1	667.62	3952.8	3.7×10^5
	0.2	163.62	83.912	245.17	14652.
	0.3	33.444	25.786	51.948	432.94
	0.4	10.424	12.050	18.397	56.715
	0.5	4.4387	7.4657	8.351	12.226
	0.6	2.4340	5.5684	3.977	2.751
	0.7	1.588	4.5506	1.376	0.132
	0.8	1.1647	3.6822	1.7×10^{-3}	3.910
	0.9	0.9084	2.223	6.474	1531.6
$m = 22$	0.1	-	-	-	-
	0.2	2.4×10^8	2.9×10^7	8.9×10^{13}	1.8×10^{14}
	0.3	2.9×10^6	4.8×10^5	1.9×10^{10}	1.6×10^{10}
	0.4	76576	17406	5.5×10^7	2.1×10^7
	0.5	4195.6	1337.5	6.4×10^5	1.3×10^5
	0.6	356.98	164.72	19833.0	2188.6
	0.7	38.757	28.654	1272.4	73.758
	0.8	5.0647	7.5355	154.6	4.2823
	0.9	1.4414	4.0612	24.42	0.6892

Notes: The local power for an ARCH(1) over different sampling frequencies
m is obtained by numerically solving the following system of equations (3.6)
and (3.7).

Table 2: Empirical sizes of CUSUM monitoring for the absolute and squared returns of a GARCH process and alternative boundaries

| | $n = 75$ | | | | $n = 125$ | | | | $n = 250$ | | | | $n = 500$ | | | |
| | LP | | HP | | LP | | HP | | LP | | HP | | LP | | HP | |
	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$
q																
					Linear Boundary in Chu et al. $b_2(t)$											
$2n$	0.185	0.124	0.221	0.170	0.176	0.108	0.186	0.124	0.156	0.107	0.171	0.132	0.146	0.095	0.167	0.136
$4n$	0.216	0.135	0.220	0.288	0.224	0.138	0.233	0.151	0.141	0.104	0.183	0.131	0.152	0.097	0.207	0.130
$6n$	0.219	0.119	0.258	0.164	0.191	0.119	0.234	0.151	0.158	0.103	0.224	0.158	0.163	0.094	0.176	0.141
$8n$	0.207	0.126	0.248	0.162	0.186	0.135	0.135	0.172	0.157	0.104	0.207	0.138	0.139	0.077	0.189	0.125
$10n$	0.229	0.133	0.263	0.186	0.178	0.125	0.125	0.161	0.179	0.096	0.206	0.129	0.151	0.103	0.182	0.126
					Linear Boundary in Zeileis et al. $b_4(t)$											
$2n$	0.269	0.276	0.362	0.346	0.280	0.223	0.323	0.288	0.229	0.217	0.305	0.285	0.244	0.214	0.274	0.274
$4n$	0.245	0.196	0.288	0.231	0.275	0.207	0.301	9.257	0.249	0.247	0.265	0.241	0.167	0.145	0.199	0.180
$6n$	0.243	0.190	0.302	0.254	0.227	0.188	0.288	0.219	0.205	0.169	0.253	0.213	0.191	0.147	0.206	0.197
$8n$	0.264	0.194	0.308	0.248	0.225	0.178	0.289	0.231	0.200	0.184	0.236	0.191	0.176	0.137	0.220	0.191
$10n$	0.267	0.197	0.307	0.264	0.230	0.190	0.278	0.230	0.229	0.174	0.246	0.213	0.160	0.133	0.234	0.218

Notes: The appreviation LP and HP refer to the Low Persistence and High Persistence GARCH specifications, respectively.

Table 3: Empirical sizes of CUSUM monitoring for the Power and Quadratic Variations based on a GARCH process and alternative boundaries

	$n = 75$				$n = 125$				$n = 250$				$n = 500$			
	LP		HP		LP		HP		LP		HP		LP		HP	
	PV_t	QV_t	PV_t	QV_t	PV_t	QV_t	PV_t	QV_t	PV_t	QV_t	PV_t	QV_t	PV_t	QV_t	PV_t	QV_t
q																
					Linear Boundary in Chu et al. $b_2(t)$											
$2n$	0.133	0.178	0.150	0.190	0.118	0.138	0.146	0.183	0.092	0.134	0.112	0.146	0.049	0.088	0.059	0.094
$4n$	0.153	0.193	0.183	0.230	0.110	0.162	0.154	0.216	0.092	0.130	0.126	0.163	0.085	0.118	0.090	0.131
$6n$	0.158	0.208	0.199		0.121	0.188	0.160	0.202	0.092	0.151	0.112	0.160	0.101	0.125	0.096	0.141
$8n$	0.137	0.197	0.214	0.261	0.124	0.177	0.165	0.249	0.128	0.188	0.134	0.192	0.105	0.157	0.113	0.165
$10n$	0.159	0.202	0.201	0.256	0.175	0.244	0.135	0.186	0.109	0.171	0.111	0.167	0.119	0.159	0.130	0.181
					Linear Boundary in Zeileis et al. $b_4(t)$											
$2n$	0.283	0.310	0.338	0.351	0.247	0.271	0.293	0.326	0.247	0.271	0.277	0.285	0.194	0.220	0.195	0.226
$4n$	0.228	0.264	0.283	0.321	0.224	0.265	0.241	0.287	0.152	0.199	0.249	0.272	0.183	0.214	0.145	0.184
$6n$	0.229	0.290	0.277	0.310	0.193	0.240	0.200	0.248	0.162	0.210	0.201	0.256	0.152	0.187	0.139	0.179
$8n$	0.197	0.264	0.276	0.333	0.185	0.224	0.262	0.303	0.134	0.194	0.183	0.218	0.134	0.170	0.150	0.193
$10n$	0.210	0.270	0.261	0.316	0.210	0.247	0.237	0.280	0.158	0.193	0.176	0.233	0.143	0.179	0.145	0.189

Notes: The appreviation LP and HP refer to the Low Persistence and High Persistence GARCH specifications, respectively.

Table 4: Empirical first hitting time of the CUSUM test for a change-point in a *low-persistent* GARCH process monitored by the *Chu et al.* (1996) linear boundary, $b_2(t)$ and *high-frequency volatility filters*

		Change in the constant of a GARCH						Change in the dynamics of a GARCH					
	PV_t												
n	$\tau = 1.1n$	1st Q	Med	3rd Q	Mean	Std	ARL	1st Q	Med	3rd Q	Mean	Std	ARL
75	83	91	97	106	106	46	23	100	110	124	129	87	46
125	138	150	156	166	160	15	22	160	169	181	176	50	38
250	275	289	295	304	298	14	23	305	314	325	317	30	43
500	550	569	577	586	578	15	28	587	598	610	600	18	50
n	$\tau = 3n$												
75	225	252	276	318	297	98	72	307	355	422	385	122	160
125	375	417	450	492	465	100	90	484	530	588	549	117	174
250	750	811	850	900	858	97	108	901	953	1014	961	91	211
500	1500	1590	1644	1702	1641	127	141	1706	1775	1844	1774	111	274
	QV_t												
n	$\tau = 1.1n$												
75	83	89	94	103	105	62	22	114	127	147	148	92	102
125	138	147	153	162	159	62	21	172	187	215	225	162	82
250	275	286	292	300	295	21	20	321	338	359	354	107	80
500	550	565	573	584	576	18	26	607	624	648	638	162	88
n	$\tau = 3n$												
75	225	241	263	296	276	97	51	350	436	574	485	177	260
125	375	403	431	468	445	114	70	545	623	725	674	208	299
250	750	791	828	875	831	135	81	983	1063	1162	1101	225	351
500	1500	1595	1614	1671	1606	206	106	1809	1917	2024	1924	208	424

Table 5: Empirical first hitting time of the CUSUM test for a change-point in a *high-persistent* GARCH process monitored by the *Chu et al.* (1996) linear boundary, $b_2(t)$ and *high-frequency volatility filters*

		Change in the constant of a GARCH						Change in the dynamics of a GARCH					
n	$\tau = 1.1n$	1st Q	Med	3rd Q	Mean	Std	ARL	1st Q	Med	3rd Q	Mean	Std	ARL
	PV_t												
75	83	94	101	118	135	128	52	97	105	119	133	119	50
125	138	152	161	178	187	133	49	157	166	178	178	89	40
250	275	293	304	318	312	85	38	302	311	324	317	28	42
500	550	574	588	603	592	31	42	586	597	609	600	20	50
	QV_t												
75	83	91	98	111	124	111	41	105	122	161	196	199	113
125	138	149	157	172	184	144	46	169	187	219	245	213	107
250	275	289	299	313	312	116	37	318	337	368	370	176	95
500	550	569	581	598	591	78	41	609	628	658	644	83	94

Table 6: Empirical first hitting time of the CUSUM test for a change-point in a *low-persistent* GARCH process monitored by the *Chu et al.* (1996) linear boundary, $b_2(t)$ and the *RiskMetrics*

RM		Change in the constant of a GARCH						Change in the dynamics of a GARCH					
n	$\tau = 1.1n$	1st Q	Med	3rd Q	Mean	Std	ARL	1st Q	Med	3rd Q	Mean	Std	ARL
75	83	100	113	139	166	166	83	115	143	224	244	230	161
125	138	159	173	198	208	152	70	179	205	250	279	249	141
250	275	303	318	343	343	164	68	332	359	399	409	259	134
500	550	303	317	342	334	75	59	627	651	683	666	99	116

Notes: For $q = 10n$ the size of the CUSUM test for the Chu et al. boundary when the control process is the RiskMetrics (RM) is: $0.355, 0.304, 0.217, 0.125$ for $n = 75, 125, 250, 500$, respectively.

Table 7: Empirical first hitting time of the CUSUM test for a change-point in a *low-persistent* GARCH process monitored by the *Zeileis et al.* (2003) linear boundary, $b_4(t)$ and *high-frequency volatility filters*

PV_t		Change in the constant of a GARCH						Change in the dynamics of a GARCH					
n	$\tau = 1.1n$	1st Q	Med	3rd Q	Mean	Std	ARL	1st Q	Med	3rd Q	Mean	Std	ARL
75		95	100	104	107	44	24	105	114	125	121	42	38
125		154	160	169	164	17	26	169	177	188	181	20	43
250		298	306	316	309	16	34	321	330	341	333	20	58
500		584	593	609	596	18	46	614	625	637	626	19	77
n	$\tau = 3n$												
75		241	264	295	274	77	50	275	309	346	316	68	91
125		393	415	445	418	92	43	443	477	520	485	83	110
250		788	825	870	820	106	70	852	893	938	893	77	144
500		1563	1609	1657	1592	150	92	1683	1699	1761	1690	119	190
QV_t													
n	$\tau = 1.1n$												
75		92	97	105	105	48	22	299	348	414	374	118	149
125		150	156	166	163	47	25	114	127	147	148	93	65
250		293	301	313	305	20	30	339	355	374	365	85	90
500		577	587	600	591	24	41	639	656	680	668	158	118
n	$\tau = 3n$												
75		236	255	280	263	85	38	475	528	593	550	133	174
125		403	428	460	433	79	58	477	526	587	546	130	171
250		775	810	851	798	145	48	899	1031	958	967	122	218
500		1546	1587	1633	1565	223	65	1696	1785	1880	1779	166	279

Table 8: Empirical first hitting time of the CUSUM test for a change-point in a *low persistence* GARCH process monitored by the *Chu et al.* (1996) linear boundary, $b_2(t)$ and *daily returns transformations*

		Change in the constant of a GARCH						Change in the dynamics of a GARCH					
n	$\tau = 1.1n$	1st Q	Med	3rd Q	Mean	Std	ARL	1st Q	Med	3rd Q	Mean	Std	ARL
	$\lvert r_t \rvert$												
75		98	111	137	147	129	64	135	175	310	283	232	200
125		154	168	192	191	106	53	187	216	271	264	175	126
250		297	310	330	318	34	43	333	358	390	371	83	96
500		579	595	617	599	28	50	618	641	670	647	42	98
	r_t^2												
75		93	104	124	139	130	57	187	319	826	446	284	363
125		150	161	182	190	138	52	229	291	436	451	378	313
250		290	304	324	323	150	49	373	415	472	482	306	207
500		571	586	607	593	34	43	657	696	746	724	236	174

Table 9: Empirical first hitting time of the CUSUM test for a change-point in a *low persistence* GARCH process monitored by the *Zeileis et al.* (2003) linear boundary, $b_2(t)$ and *daily returns transformations*

		Change in the constant of a GARCH						Change in the dynamics of a GARCH					
n	$\tau = 1.1n$	1st Q	Med	3rd Q	Mean	Std	ARL	1st Q	Med	3rd Q	Mean	Std	ARL
	$\lvert r_t \rvert$												
75		103	115	134	131	70	48	155	188	255	256	182	173
125		155	168	186	187	109	49	194	213	245	228	69	89
250		310	325	346	331	28	56	352	371	398	379	39	104
500		601	618	638	621	31	72	654	675	700	679	37	129
	r_t^2												
75		97	108	127	128	81	45	130	151	188	187	122	104
125		162	176	195	188	67	50	220	252	30	296	169	158
250		299	315	337	323	35	48	386	416	454	437	131	162
500		586	604	626	612	40	62	695	728	769	746	171	196

Table 10: Empirical sizes of CUSUM monitoring for half-daily processes of a GARCH model for alternative boundaries

GARCH	$n = 75$		$n = 125$		$n = 250$		$n = 500$	
	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2
Chu et al. boundary, $b_2(t)$								
Normal	0.154	0.220	0.145	0.180	0.097	0.160	0.085	0.117
t(6)	0.212	0.283	0.157	0.221	0.140	0.189	0.130	0.153
Zeileis et al. boundary, $b_4(t)$								
Normal	0.215	0.265	0.194	0.228	0.148	0.184	0.118	0.145
t(6)	0.318	0.386	0.282	0.323	0.252	0.284	0.240	0.254

Table 11: Empirical first hitting time of the CUSUM test for a change-point in the constant of a low persistent GARCH process monitored by:

(i) ARLs (in days) of squared and absolute returns

	Chu et al. boundary $b_2(t)$								Zeileis et al. boundary $b_4(t)$							
	Normality				Student's t(6)				Normality				Student's t(6)			
	Daily		Half-daily		Daily		Half-daily		Daily		Half-daily		Daily		Half-daily	
n	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2	$|r_t|$	r_t^2
75	64	57	33	38	69	64	77	87	48	44	35	36	52	50	61	69
125	51	55	30	34	47	45	73	104	47	49	33	34	48	44	60	82
250	47	49	30	34	44	52	49	94	56	52	42	38	57	55	57	87
500	50	43	37	34	51	47	61	140	72	62	57	51	74	65	75	132

(ii) ARLs (in days) of high-frequency volatility filters

	Chu et al. boundary $b_2(t)$				Zeileis et al. boundary $b_4(t)$			
	Student's t(6)				Student's t(6)			
	Daily		Half-daily		Daily		Half-daily	
n	PV_t	QV_t	PV_t	QV_t	PV_t	QV_t	PV_t	QV_t
75	26	26	57	67	24	24	42	51
125	21	21	44	81	25	25	47	70
250	22	20	36	76	34	30	46	71
500	27	24	42	91	44	39	62	97

Notes: We consider the low persistent GARCH process generated at the 30-minute frequency. The change point is at $\tau = 1.1n$ and the constant of the GARCH increases by three. Two alternative monitoring frequencies are considered: the daily and half-daily frequencies for the following categories of processes: (i) Squared and absolute returns, r_t^2 and $|r_t|$, respectively, and (ii) High frequency volatility filters, the Power and Quadratic Variation, PV_t and QV_t, respectively.

Table 12: ARL for the empirical first hitting time of the CUSUM test for a change-point in a GARCH
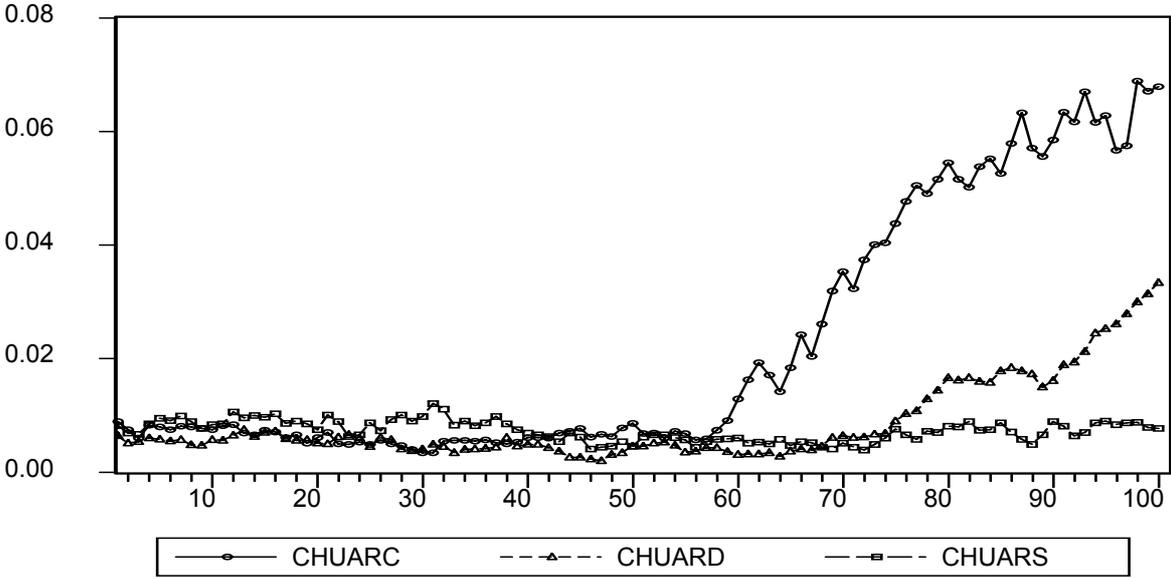
$n = 250$ ARL:	Chu et al.		Zeileis et al.	
	$|r_t|$	r_t^2	$|r_t|$	r_t^2
Constant change-point	79	76	74	67
Dynamic change-point	175	351	141	219
	PV_t	QV_t	PV_t	QV_t
Constant change-point	44	37	44	38
Dynamic change-point	82	145	77	118

Table 13: Prediction probability value and detection delay of the sequential predictive test for a change-point in a GARCH process

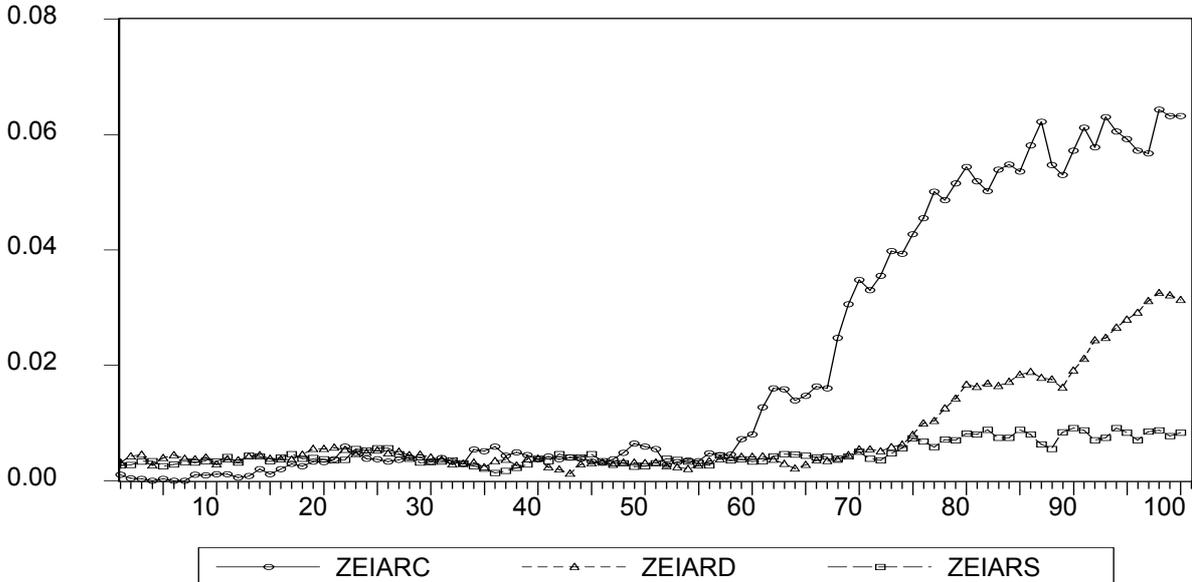| Boundaries	Monitoring	5-days ahead				10-days ahead			
	Process	$\max p_0$	p_1	τ_p	ARL	$\max p_0$	p_1	τ_p	ARL
		Constant change-point							
Chu et al.	$|r_t|$	0.0087	0.0102	58	8	0.0121	0.0290	60	10
	r_t^2	0.0145	0.0164	62	12	0.0145	0.0173	53	3
Zeileis et al.	$|r_t|$	0.0087	0.0113	63	13	0.0091	0.0127	61	11
	r_t^2	0.0085	0.0191	64	14	0.0155	0.0177	56	6
		Dynamic change-point							
Chu et al.	$|r_t|$	0.0087	0.0090	84	34	0.0121	0.0129	78	28
	r_t^2	0.0145	-	-		0.0145	-	-	
Zeileis et al.	$|r_t|$	0.0087	0.092	84	34	0.0091	0.0099	76	26
	r_t^2	0.0085	0.0086	94	44	0.0155	-	-	
		Constant change-point							
Chu et al.	PV_t	0.0085	0.0111	58	8	0.0138	0.0173	59	9
	QV_t	0.0115	0.0122	58	8	0.0155	0.0167	56	6
Zeileis et al.	PV_t	0.0077	0.0101	60	10	0.0157	0.0183	57	7
	QV_t	0.0088	0.0108	58	8	0.0155	0.0162	60	10
		Dynamic change-point							
Chu et al.	PV_t	0.0085	0.0103	70	20	0.0138	0.0145	77	27
	QV_t	0.0115	0.0125	93	43	0.0155	0.0158	82	32
Zeileis et al.	PV_t	0.0077	0.0089	71	21	0.0157	0.0162	78	28
	QV_t	0.0088	0.0104	91	41	0.0155	0.0162	82	32

Notes: Under H_1 of a change point in the constant of the low persistent GARCH the historical sample size is $n = 250$, the time of the break is $\tau = 1.5n$, and the starting point for the predictive exercise is $q_0 = 1.3n$. τ_p is the first signal of change, $\max p_0$ is the maximum prediction probability value under H_0, p_1 is the first sequential prediction probability value under H_1.

Figure 1: Probabilities of the prediction-based CUSUM for the absolute returns
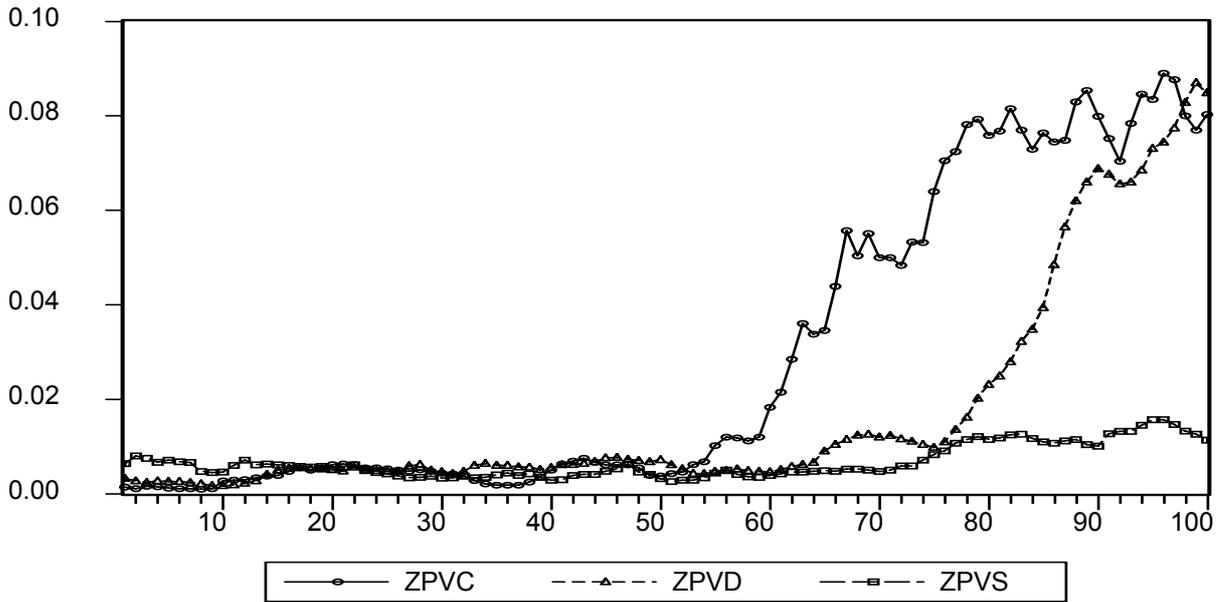and the Chu et al. boundary.

Notes: The three time series of the prediction probabilities refer to a change in the constant (CHUARC), a change in the dynamics (CHUARD) and no break (CHUARS). The Chu et al. (1996) boundary is used. The results are valid for a 10-day forecasting horizon and the true change point occurs at observation 50.

Figure 2: Probabilities of the prediction-based CUSUM for the absolute returns
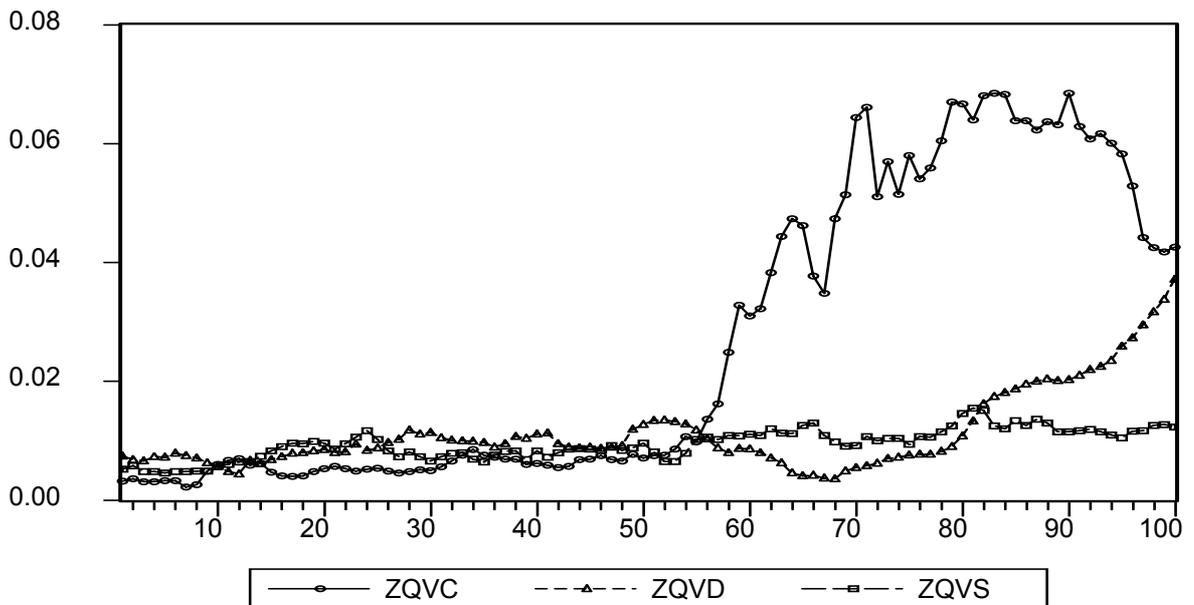and the Zeileis et al. boundary.

Notes: The three time series of the prediction probabilities refer to a change in the constant (ZEIARC), a change in the dynamics (ZEIARD) and no break (ZEIARS). The Zeileis et al. (2004) boundary is used. The results are valid for a 10-day forecasting horizon and the true change point occurs at observation 50.

Figure 3: Probabilities of the prediction-based CUSUM for the Power Variation (PV)
and the Zeileis et al. boundary.

Notes: The three time series of the prediction probabilities refer to a change in the constant (ZPVC), a
change in the dynamics (ZPVD) and no break (ZPVS). The Zeileis et al. (2004) boundary is used. The results
are valid for a 10-day forecasting horizon and the true change point occurs at observation 50.

Figure 4: Probabilities of the prediction-based CUSUM for the Quadratic Variation (QV)
and the Zeileis et al. boundary.

Notes: The three time series of the prediction probabilities refer to a change in the constant (ZQVC), a
change in the dynamics (ZQVD) and no break (ZQVS). The Zeileis et al. (2004) boundary is used. The results
are valid for a 10-day forecasting horizon and the true change point occurs at observation 50.