Price Discovery
in a Matching and Bargaining Market
with Aggregate Uncertainty

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In a market where buyers and sellers are strategic and \textbf{uncertain} about demand and supply, at what price should they trade?

Study dynamic market with \textbf{search frictions} and \textbf{decentralized bilateral bargaining}

- e.g. second-hand housing market, used car market, labor market

2 states:

- $H$: high-demand low-supply (sellers’ market)
- $L$: high-supply low-demand (buyers’ market)

Traders learn from search experiences

If search frictions are small, would the transaction prices be close to the true-state Walrasian (or competitive, or market-clearing) price?
Main Results

In our model, as search frictions converge to 0, the market discovers the true-state Walrasian price quickly:

- transaction prices converge to the true-state Walrasian price in expectation
- the rate of convergence is linear in search frictions, the same as it would be if the state were commonly known
Literature (Dynamic matching and bargaining games)

- Initiated by Rubinstein & Wolinsky (1985), homogeneous buyers/sellers, no uncertainty
- Heterogeneous buyers/sellers, complete info bargaining
  - Gale (1987), Mortensen & Wright (2002)
- Heterogeneous buyers/sellers, IPV bargaining
- Common values uncertainty
- Aggregate (demand-supply) uncertainty
  - Majumdar, Shneyerov, & Xie (2016), Lauermann, Merzyn, & Virag (2018)
Model

- Buyers/sellers arrive at market deterministically and continuously
- Each seller has a unit supply of a homogeneous, indivisible good; cost is 0
- Each buyer has a unit demand; valuation is 1
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- Each buyer has a unit demand; valuation is 1
- Two possible states: $\omega \in \{H, L\}$; inflow rates of buyers/sellers in state $\omega$ are $\lambda_B^\omega$ and $\lambda_S^\omega$

**Assumption 1:** $\lambda_B^H > \lambda_S^H$ and $\lambda_B^L < \lambda_S^L$.

- State is constant over time. No one knows the true state; common prior belief $\phi^\omega$

**Note:** flow Walrasian price is 1 if $\omega = H$ and 0 if $\omega = L$
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- Every trader is risk neutral
- Continuous time, infinite horizon; focus on steady state
• Given stocks of buyers/sellers $\Lambda_B, \Lambda_S$, the mass of pairs matched per unit time is $\mu \cdot \min\{\Lambda_B, \Lambda_S\}$

• Who gets matched and Who matches whom are random

• Once matched, they bargain:
  1. Nature randomly chooses a proposer: buyer with prob. $\beta_B \in (0, 1)$; seller with prob. $\beta_S \equiv 1 - \beta_B$
  2. Proposer makes take-it-or-leave-it price offer
  3. Responder chooses to accept or reject
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**Assumption 2**: Upon meeting, each trader observes the total time his partner has participated in the market.

- If trade at $p$, buyer leaves with payoff $1 - p$, seller leaves with $p$
- If don’t trade, stay searching for another match

**Friction profile**: $(r, \delta)$
- $\delta > 0$: exogenous exit rate
- $r \geq 0$: time discount rate
Full trade (steady state) market equilibrium

Basic equilibrium objects:

- steady state stocks and distributions of traders
- traders’ beliefs about state
- traders’ bargaining strategies
Full trade (steady state) market equilibrium

Basic equilibrium objects:

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such that

- Given bargaining strategies, steady state equations are satisfied to maintain the stocks and distributions
- Given steady state stocks and distributions, the traders’ beliefs and bargaining strategies constitute Perfect Bayesian Equilibrium
- In addition, restrict attention to full trade equilibria (FTE), in which every meeting on equilibrium path results in trade.
Steady state stocks

For each $\omega = L, H$, stocks $\Lambda^\omega_B, \Lambda^\omega_S$ satisfy

$$\lambda^\omega_B = \delta \Lambda^\omega_B + \mu \min\{\Lambda^\omega_B, \Lambda^\omega_S\}$$

$$\lambda^\omega_S = \delta \Lambda^\omega_S + \mu \min\{\Lambda^\omega_B, \Lambda^\omega_S\}$$

so that

$$\Lambda^\omega_B = \frac{(\delta + \mu) \lambda^\omega_B - \mu \min\{\lambda^\omega_B, \lambda^\omega_S\}}{\delta (\delta + \mu)},$$

$$\Lambda^\omega_S = \frac{(\delta + \mu) \lambda^\omega_S - \mu \min\{\lambda^\omega_B, \lambda^\omega_S\}}{\delta (\delta + \mu)}.$$

Note: $\Lambda^H_B > \Lambda^H_S$ and $\Lambda^L_B < \Lambda^L_S$. 
Steady state finding rates

For each $\omega = L, H$, finding rates $\alpha^\omega_B, \alpha^\omega_S$ are

$$\alpha^\omega_B \equiv \frac{\mu \min\{\Lambda^\omega_B, \Lambda^\omega_S\}}{\Lambda^\omega_B}, \quad \alpha^\omega_S \equiv \frac{\mu \min\{\Lambda^\omega_B, \Lambda^\omega_S\}}{\Lambda^\omega_S}$$
Steady state finding rates

For each \( \omega = L, H \), finding rates \( \alpha_B^\omega, \alpha_S^\omega \) are

\[
\alpha_B^\omega \equiv \frac{\mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}}{\Lambda_B^\omega}, \quad \alpha_S^\omega \equiv \frac{\mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}}{\Lambda_S^\omega}
\]

In particular, short sides’ finding rates are

\[
\alpha_B^L = \alpha_S^H = \mu,
\]

long sides’ finding rates are

\[
\alpha_B^H = \frac{\delta \mu \lambda_S^H}{(\delta + \mu) \lambda_B^H - \mu \lambda_S^H} < \mu,
\]

\[
\alpha_S^L = \frac{\delta \mu \lambda_B^L}{(\delta + \mu) \lambda_S^L - \mu \lambda_B^L} < \mu.
\]

Lemma 1. \( \alpha_B^H \) and \( \alpha_S^L \) are \( O(\delta) \).
Steady state distributions

- Let $G^\omega_B(t_B)$ be the fraction of buyers’ steady-state stock in state $\omega$ who have been in the market for less than time $t_B$.

- Steady-state equation for $G^\omega_B(\cdot)$ implies:

$$G^\omega_B(t_B) = 1 - \exp\left(- (\delta + \alpha^\omega_B) t_B \right)$$
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**Alternative Interpretation:** conditional distribution of searching time

- $G^\omega_B(t_B)$ is, from an unmatched buyer’s perspective, the prob. of being matched after some searching time less than $t_B$, conditional on the event that the true state is $\omega$ and this buyer will meet a seller (rather than exogenously exit before meeting)

- Similar note for $G^\omega_S(t_S) = 1 - \exp(-\left(\delta + \alpha^\omega_S\right)t_S)$
Belief formation
Search history and bargaining history

**Search history** (on or off equilibrium path) of a buyer who has met $n$ sellers:

$$(t_{B1}, \ldots, t_{Bn}, t_{B(n+1)}; t_{S1}, \ldots, t_{Sn})$$

- $t_{Bi}$ for $i \in \{1, \ldots, n\}$ is searching time spent to have the $i$-th meeting
- $t_{Si}$ for $i \in \{1, \ldots, n\}$ is the observed time on the market of the $i$-th seller met
- $t_{B(n+1)}$ is the time on the market since last meeting
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**Bargaining history:**

- which side proposed in previous meetings
- previous price offers
- that these offers are rejected
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Can WLOG assume every trader only uses search history to update belief, since focus on FTE.
Belief formation
Updating from search history

\[ h_B \equiv (t_{B1}, \ldots, t_{Bn}, t_{B(n+1)}; t_{S1}, \ldots, t_{Sn}) \]

- Given \( \alpha_B^\omega, \alpha_S^\omega, G_B^\omega(t_B), G_S^\omega(t_S) \), a buyer’s belief \( \pi_B^\omega(h_B) \) about state \( \omega \) after \( h_B \) can be computed from Bayes’ rule.
- \( \pi_B^\omega(h_B) \) depends on \( h_B \) only through \( \sum_{i=1}^{n+1} t_{Bi} \equiv t_B, \sum_{i=1}^{n} t_{Si} \equiv t_S \) and \( n \).
Belief formation
Updating from search history

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- \( \pi^\omega_B(h_B) \) depends on \( h_B \) only through \( \sum_{i=1}^{n+1} t_{Bi} \equiv t_B, \sum_{i=1}^{n} t_{Si} \equiv t_S \) and \( n \)
- Similarly, \( \pi^\omega_S(h_S) \) depends on \( h_S \) only through \( \sum_{i=1}^{n} t_{Bi} \equiv t_B, \sum_{i=1}^{n+1} t_{Si} \equiv t_S \) and \( n \)
- Write \( \pi^\omega_B(t_B, t_S, n) \) and \( \pi^\omega_S(t_B, t_S, n) \)
Belief formation
Updating from search history

\[ h_B \equiv (t_{B1}, \ldots, t_{Bn}, t_{B(n+1)}; t_{S1}, \ldots, t_{Sn}) \]

- Given \( \alpha^\omega_B, \alpha^\omega_S, G_B^\omega(t_B), G_S^\omega(t_S) \), a buyer’s belief \( \pi^\omega_B(h_B) \) about state \( \omega \) after \( h_B \) can be computed from Bayes’ rule.
- \( \pi^\omega_B(h_B) \) depends on \( h_B \) only through \( \sum_{i=1}^{n+1} t_{Bi} \equiv t_B, \sum_{i=1}^{n} t_{Si} \equiv t_S \) and \( n \).
- Similarly, \( \pi^\omega_S(h_S) \) depends on \( h_S \) only through \( \sum_{i=1}^{n} t_{Bi} \equiv t_B, \sum_{i=1}^{n+1} t_{Si} \equiv t_S \) and \( n \).
- Write \( \pi^\omega_B(t_B, t_S, n) \) and \( \pi^\omega_S(t_B, t_S, n) \).

**Feature:** \( \pi^\omega_B(t_B, t_S, 1) = \pi^\omega_S(t_B, t_S, 1) \) for every \( t_B, t_S \).
- meeting on eqm path is the first meeting for both.
- bargaining on eqm path is under sym info.
Bellman equations

- Bargaining strategies are fully characterized by the continuation payoffs (or search values) $W_B(h_B)$ and $W_S(h_S)$ just after breaking-up.
- Write $W_B(t_B, t_S, n)$ and $W_S(t_B, t_S, n)$.
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- Write $W_B(t_B, t_S, n)$ and $W_S(t_B, t_S, n)$.

Let $T_B, T_S$ be independent r.v. that follow distributions $G_B^\omega(\cdot)$, $G_S^\omega(\cdot)$.

$$W_B(t_B, t_S, n) = \sum_{\omega=L,H} \pi_B^\omega(t_B, t_S, n) \frac{\alpha_B^\omega}{\delta + \alpha_B^\omega} \mathbb{E}[e^{-rT_B} q_B(t_B + T_B, t_S, n; T_S)|\omega]$$

where $q_B(t_B + T_B, t_S, n; T_S) \equiv \beta_B \max \{1 - W_S(t_B + T_B, T_S, 1), W_B(t_B + T_B, t_S + T_S, n + 1)\} + \beta_S \max \{W_B(t_B + T_B, T_S, 1), W_B(t_B + T_B, t_S + T_S, n + 1)\}$

Similarly for $W_S(t_B, t_S, n)$.
Given $\alpha_B^\omega, \alpha_S^\omega, G_B^\omega(\cdot), G_S^\omega(\cdot), \pi_B^\omega(\cdot), \pi_S^\omega(\cdot)$ derived above, full trade (market) equilibrium (FTE) can be redefined as functions

$$W_B, W_S : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N} \to [0, 1]$$

that solve buyers’ and sellers’ Bellman equations and such that the trading condition

$$W_B(t_B, t_S, 1) + W_S(t_B, t_S, 1) \leq 1$$

holds for every $(t_B, t_S)$.

Transaction prices on equilibrium path are:
- either $W_S(t_B, t_S, 1)$ when buyer proposes
- or $1 - W_B(t_B, t_S, 1)$ when seller proposes
Suppose true state \( \omega \) is commonly known \( (\phi^\omega = 1) \).

- \( W_B, W_S \) become constants

\[
\overline{W}_B^\omega = \frac{\beta_B \alpha_B^\omega}{r + \delta + \beta_B \alpha_B^\omega + \beta_S \alpha_S^\omega},
\]
\[
\overline{W}_S^\omega = \frac{\beta_S \alpha_S^\omega}{r + \delta + \beta_B \alpha_B^\omega + \beta_S \alpha_S^\omega}.
\]

- \( \overline{W}_B^\omega + \overline{W}_S^\omega < 1 \)
- \( \overline{W}_B^H, 1 - \overline{W}_S^H, 1 - \overline{W}_B^L, \overline{W}_S^L = O(r + \delta) \)
  - because \( \alpha_B^L = \alpha_S^H = \mu \) and \( \alpha_B^H, \alpha_S^L = O(\delta) \)
Proposition 1. If true state $\omega$ is commonly known,

- $\forall (r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, $\exists$ a unique FTE.
- $\exists C_0, C_1 > 0$, not depending on $r, \delta$, s.t. when $r + \delta > 0$ is sufficiently small,

$$C_0 \cdot (r + \delta) \leq \frac{1 - W_B^H}{1 - W_B^L} \leq C_1 \cdot (r + \delta),$$

i.e., discrepancy between equilibrium transaction prices and Walrasian price is of order $r + \delta$. 

Existence, uniqueness, rate of convergence under certainty
Uniqueness

Return to the aggregate uncertainty case \((\phi^L, \phi^H \in (0, 1))\)

- Neglect the trading condition: FTE candidate defined only by a pair of Bellman equations

**Proposition 2 (Uniqueness).** \(\forall (r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++},\) there is at most one FTE.

**Sketch of proof:** Apply Contraction Mapping Theorem to show that the system of Bellman equations has a unique solution.
Proposition 3. In any FTE,

- \( \pi_L^B(t_B, t_S, n) \) and \( W_B(t_B, t_S, n) \) are continuous in \((t_B, t_S)\),
  nonincreasing in \( t_B \), and nondecreasing in \( t_S \);
- \( \pi_H^S(t_B, t_S, n) \) and \( W_S(t_B, t_S, n) \) are continuous in \((t_B, t_S)\),
  nondecreasing in \( t_B \), and nonincreasing in \( t_S \);
- \( \forall (t_B, t_S, n) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}, \)
  \[
  W_H^B \leq W_B(t_B, t_S, n) \leq W_L^B,
  \]
  \[
  W_L^S \leq W_S(t_B, t_S, n) \leq W_H^S.
  \]
Belief convergence

- Traders’ bargaining values (on equilibrium path) depend on their outside option values.
- Their outside option values depend on their first-order beliefs and their bargaining values of off-equilibrium future bargaining.
- Values of off-equilibrium future bargaining depend on second-level outside option values, which in turn depend on second-order beliefs and bargaining values of second-level off-equilibrium future bargaining; and so on.
- In a off-equilibrium bargaining, buyer and seller do not have symmetric info; one or both of their beliefs are formed based on wrong info about $n$.
- However, all these on- and off-equilibrium beliefs become asymptotically precise in expectation.
Belief convergence

Let $T_{Bi}$’s and $T_{Si}$’s be *independent* random copies of $T_B$ and $T_S$ respectively.

**Lemma 3.** For $j = B, S$,

$$\max_{1 \leq k_1, k_2, k_3 \leq n} \left\{ \mathbb{E} \left[ \pi_{j}^L \left( \sum_{i=1}^{k_1} T_{Bi}, \sum_{i=1}^{k_2} T_{Si}, k_3 \right) \mid H \right] \right\} \leq (c_1 + c_2 n) \cdot \delta,$$

$$\max_{1 \leq k_1, k_2, k_3 \leq n} \left\{ \mathbb{E} \left[ \pi_{j}^H \left( \sum_{i=1}^{k_1} T_{Bi}, \sum_{i=1}^{k_2} T_{Si}, k_3 \right) \mid L \right] \right\} \leq (c_1 + c_2 n) \cdot \delta,$$

where $c_1, c_2$ are constants not depending on $r, \delta, n$. 
Intuition:

- Say true state is $H$, and let $\delta \to 0$.
- Recall that $\alpha^H_S = \mu$ but $\alpha^H_B = O(\delta)$.
- Buyers’ random searching time $T_B \to \infty$ in probability, but $T_S$ does not.
- The reverse is true if true state is $L$.
- Realizations of $T_B$, $T_S$ are more and more informative as $\delta \to 0$. 
Convergence of prices
To no uncertainty benchmark

Proposition 4. In any FTE,

\[ 0 \leq \begin{align*}
    \mathbb{E}[W_B(T_B, T_S, 1)|H] - W_B^H, \\
    W_B^H - \mathbb{E}[W_S(T_B, T_S, 1)|H], \\
    \mathbb{E}[W_S(T_B, T_S, 1)|L] - W_S^L
\end{align*} \leq C \cdot \delta, \]

where \( C \) is a constant that does not depend on \( r, \delta \).

- Convergence in expectation (Recall that \( \forall (t_B, t_S) \)
  \( W_B^H \leq W_B(t_B, t_S, 1) \leq W_B^L \) and \( W_S^L \leq W_S(t_B, t_S, 1) \leq W_S^H )

- expected discrepancy between equilibrium transaction prices and true-state no uncertainty benchmark price is of order \( \delta \).
Convergence of prices
To true-state Walrasian price

Main Theorem: \( \exists \) constants \( C_0, C_1 > 0 \) not depending on \( r, \delta \) s.t. if \( r + \delta > 0 \) is sufficiently small, any FTE satisfies

\[
C_0 \cdot (r + \delta) \leq 
\begin{align*}
\mathbb{E} [W_B(T_B, T_S, 1) \mid H], \\
1 - \mathbb{E} [W_S(T_B, T_S, 1) \mid H], \\
1 - \mathbb{E} [W_B(T_B, T_S, 1) \mid L], \\
\mathbb{E} [W_S(T_B, T_S, 1) \mid L],
\end{align*}
\leq C_1 \cdot (r + \delta),
\]

i.e., expected discrepancy between equilibrium transaction prices and the true-state Walrasian price is of order \( r + \delta \).
**Existence**

**Proposition 5.** \( \forall r > 0, \exists \delta > 0 \) s.t. whenever \( r \geq r \) and \( 0 < \delta \leq \delta \), the FTE candidate satisfies

\[
W_B(t_B, t_S, 1) + W_S(t_B, t_S, 1) \leq 1 \quad \forall (t_B, t_S) \in \mathbb{R}_+ \times \mathbb{R}_+.
\]

**Corollary 3.** For any level \( \tau > 0 \), \( \exists (r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++} \) with \( r + \delta = \tau \) s.t. a FTE exists under \((r, \delta)\).
Summary

- Study dynamic model of a market with search friction and bilateral random-proposer take-it-or-leave-it bargaining
- Two possible states:
  - at $H$ state, more buyers than sellers
  - at $L$ state, more sellers than buyers
- The only info transmitted in a meeting is the time a trader spent on the market
- As search frictions vanish, the market discovers the true-state competitive price quickly
  - Transaction prices converge to the true-state Walrasian price in expectation
  - Rate of convergence is linear in the total search friction, the same as it would be if the state were commonly known.